The GNSS Observation Equation

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1 Introduction

Contributed thoughts by (Ramakrishna, 2011) and (Elburg, 2011) seem to indicate that the observation equations used since about thirty years in the scientific (and other high accuracy) exploitation of the Global Navigation Satellite Systems (GNSS) contain gross errors. In our brief review we show that this is not true.

The differences between the observation equations actually used, and well documented, e.g., in Beutler (2005), and the above mentioned alternatives are caused by basically the same incorrect interpretation of the GNSS observation equation by (Ramakrishna, 2011) and (Elburg, 2011).

In Section 2 we reproduce the slightly simplified, but otherwise correct GNSS observation equations. In Section 3 we derive a simplified version of these equations, which is identical with the original equations up and including the terms of first order in the flight times of the signals. The result may now be easily compared to the alternative formulations in Section 4. In Section 5 we draw the conclusions.

2 The GNSS Observation Equations

According to (Beutler, 2005), Vol. I, Eq. (8.85) the non-relativistic version of the GNSS observation equations for the GNSS code observable (C/A- (clear access) or P- (precise) code) read as:

$$p_i^j = \rho_i^j - c\,\Delta t^j + c\,\Delta t_i + \Delta\rho_{i_{\text{ion}}}^j + \Delta\rho_{i_{\text{trop}}}^j + \epsilon_{i_{\text{cod}}}^j\ , \tag{1}$$

where

- c is the speed of light.
- $p_i^j = c\left((t_i + \Delta t_i) - (\tau^j + \Delta t^j)\right)$ the so-called pseudorange, the difference between the clock readings at reception of the signal at the receiver and emission of the signal at the satellite, respectively.
- j is the satellite index, i the receiver index.
- $\rho_i^j \doteq |\boldsymbol{r}(\tau^j) - \boldsymbol{R}(t_i)|$, is the geometric distance between the satellite at signal emission time τ^j and the receiver at signal reception time t_i (ρ_i^j also is referred to as *slant range* between satellite and receiver),
- Δt^j is the offset of the satellite clock w.r.t. GNSS time at emission time,
- Δt_i the offset of the receiver clock w.r.t. GNSS time at signal reception time;
- $\Delta\rho_{i_{\text{trop}}}^j$ is the tropospheric range correction,
- $\Delta\rho_{i_{\text{ion}}}^j$ the ionospheric range correction, and
- $\epsilon_{i_{\text{cod}}}^j$ the measurement error of the observation.

$t_i + \Delta t_i$ and $\tau^j + \Delta t^j$ are the readings of the receiver clock and of the satellite clock, at the reception and emission of one and the same signal, respectively.

For common view time transfer using the code observation equations one assumes $\Delta t^j = 0$. In practice this is equivalent to adopting the GNSS satellite clock corrections to GNSS time from the International GNSS Service (IGS).

In scientific applications, when “hunting the mm and the ps (picosecond)” in the receiver positions and clock corrections, respectively, the ionospheric correction, the tropospheric range correction and the error term have to be studied carefully. One also would have to include the phase observation equation. Here we are “hunting” differences between the correct and the alternative approaches of the order of 60 ns corresponding to about 20 m. It is therefore perfectly allowed to assume an Earth without atmosphere (!) and satellite clocks perfectly synchronized to GNSS time. The simplified observation equation then read as:

$$p_i^j = \rho_i^j + c\,\Delta t_i\ , \tag{2}$$

where

$$p_i^j = c\left(t_i + c\,\Delta t_i - \tau^j\right)\ .$$

ρ_i^j has the same definition as in Eq. (1).

In GNSS data analysis using real data one has to account for special and general relativistic effects. This implies in particular that the parameterized post-Newtonian (PPN) version of the equations of motion has to be used to derive the satellite trajectory $\boldsymbol{r}(t)$. The relativistic effects in the satellite clock are approximately dealt with by applying a common frequency offset to all satellite clocks. They also cause a small time-varying term in the observation equations (1) and (2). As these effects are very small, numerically, and have nothing whatsoever to do with the problem treated here we simply refer the interested reader to (Petit and Luzum, 2010), Sections 10.2 and 10.3, for more information. Using Eqs. (2) we obtain the result of the common view time-synchronization for two receivers $i = 1, 2$ using one and the same satellite j:

$$\Delta t_1 - \Delta t_2 = \frac{1}{c} \left[p_1^j - p_2^j - \left(\rho_1^j - \rho_2^j \right) \right] \tag{3}$$

Our result is consistent with Lombardi et al (2001).

3 Approximate Version of the GNSS Observation Equations

Despite the fact that there is no necessity whatsoever to do that, we approximate the propagation paths as follows:

$$\begin{aligned} \rho_i^j &\doteq |\boldsymbol{r}(\tau^j) - \boldsymbol{R}(t_i)| \\ &= |\boldsymbol{r}(t_i - \rho_i^j/c) - \boldsymbol{R}(t_i)| \\ &\approx |\boldsymbol{r}(t_i) - \rho_i^j/c \cdot \dot{\boldsymbol{r}}(t_i) - \boldsymbol{R}(t_i)| \\ &= |\boldsymbol{r}(t_i) - \boldsymbol{R}(t_i) - \rho_i^j/c \cdot \dot{\boldsymbol{r}}(t_i)| \\ &\doteq |\boldsymbol{\rho}_{i0}^j - \rho_i^j/c \cdot \dot{\boldsymbol{r}}(t_i)| \\ &\approx \left[(\rho_{i0}^j)^2 - 2\, \rho_i^j/c\, (\boldsymbol{\rho}_{i0}^j(t_i) \cdot \dot{\boldsymbol{r}}(t_i)) \right]^{1/2} \\ &\approx \rho_{i0}^j \cdot \left[1 - 2/(\rho_{i0}^j \cdot c)\, (\boldsymbol{\rho}_{i0}^j \cdot \dot{\boldsymbol{r}}(t_i)) \right]^{1/2} \\ &\approx \rho_{i0}^j - \frac{\boldsymbol{\rho}_{i0}^j \cdot \dot{\boldsymbol{r}}(t_i)}{c} \end{aligned} \tag{4}$$

As in Eq. (1) the receiver index is i and the satellite index j. $\boldsymbol{\rho}_{i0}^j \doteq \boldsymbol{r}(t_i) - \boldsymbol{R}(t_i)$ is the topocentric vector of the satellite referring to observation time t_i of station i. In the transition from Eq. $(4)_6$ to $(4)_7$ we used the approximation $\rho_{i0}^j = \rho_i^j$, which is correct up to the terms of the first order in ρ_i^j.

4 The GNSS Observation Equations and Proposed Alternatives

We easily see that Eq. $(4)_8$ corresponds to Eqs. (4) and (5) in Ramakrishna (2011): If we divide the left- and right-hand sides of EQ.$(4)_8$ by the speed of light c, it becomes identical with the above mentioned equations in Ramakrishna (2011) – apart from the pre-factor $1/\sqrt{1 - v^2/c^2}$, which was recognized by (Ramakrishna, 2011) as irrelevant for "our" problem.

5 Conclusions

1. The analysis in Ramakrishna (2011) is basically correct.
2. The analysis is, however, based on wrong assumptions: No GNSS receiver manufacturer and no GNSS analyst striving to achieve the highest possible accuracy with GNSS would use Eq. (1) in Ramakrishna (2011) as observation equations. A serious analysis has to start from equations of motion of the type (1) in our treatment of the problem.
3. The analysis by (Elburg, 2011) is based on the same basic mis-interpretation of the GNSS observation equation.
4. **The issue raised by (Elburg, 2011) and (Ramakrishna, 2011) therefore is a non-issue**.

References

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