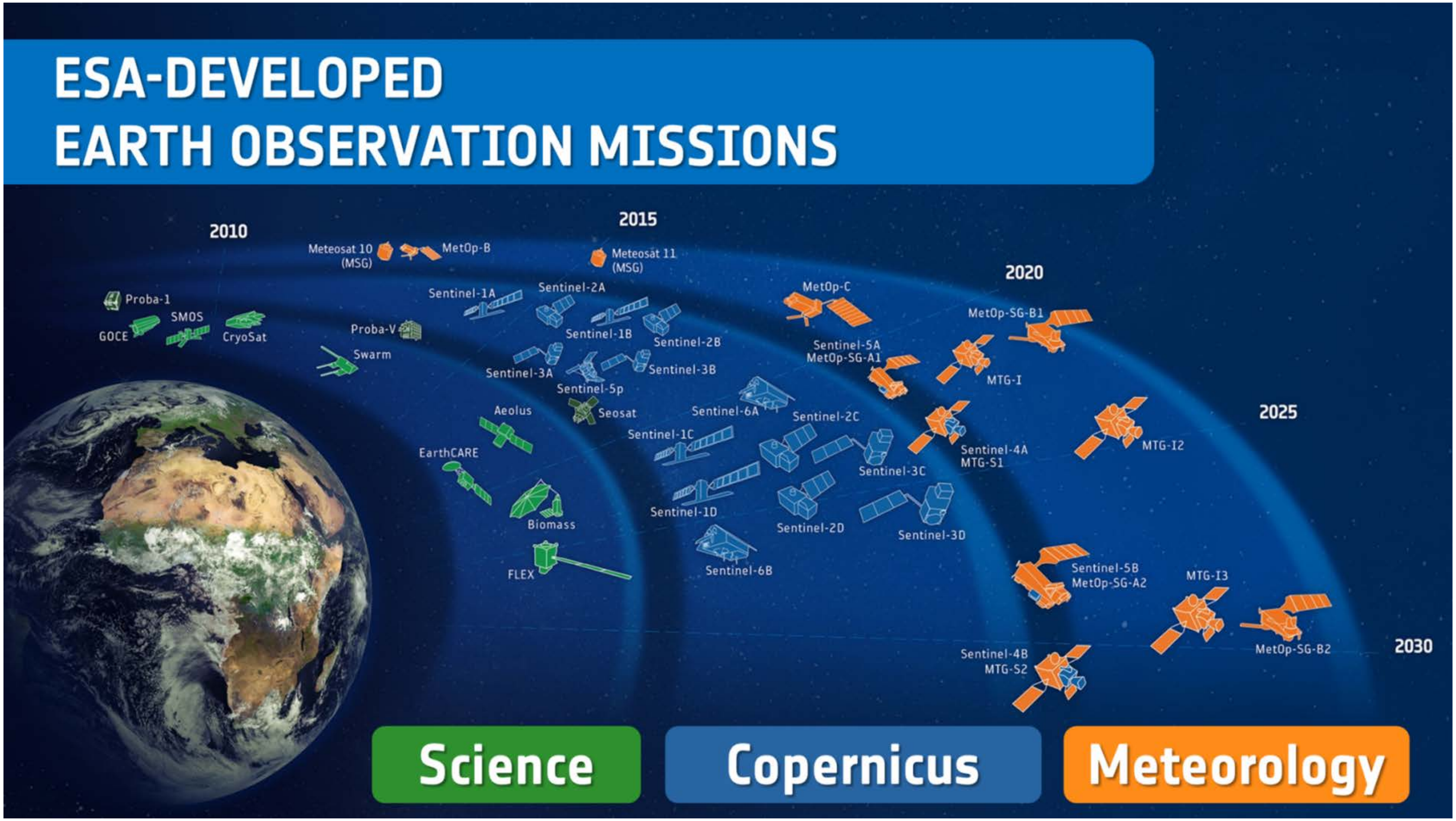
Precise Orbit Determination

Adrian Jäggi

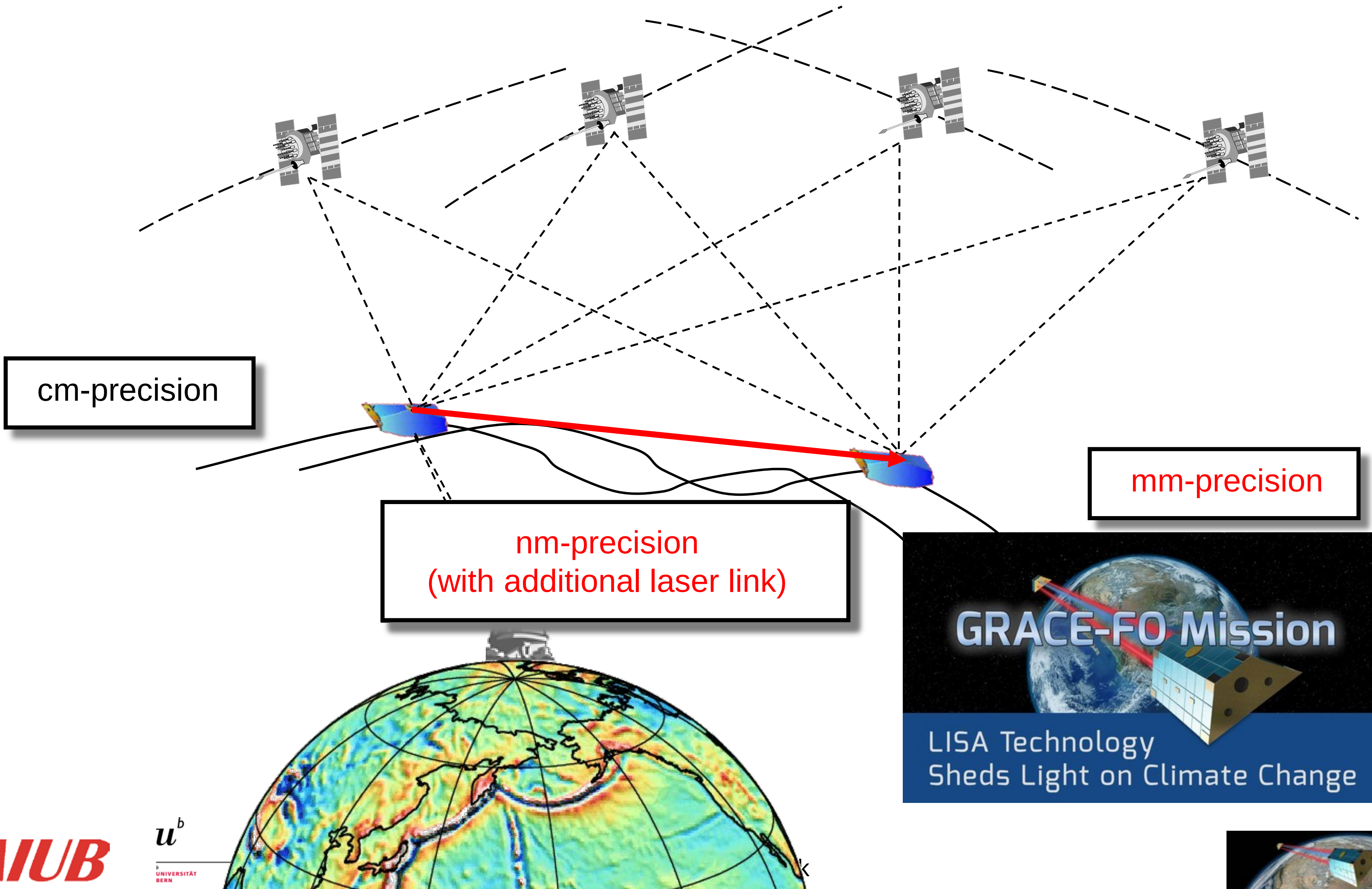
Astronomical Institute
University of Bern

Lecture Contents

A multitude of Earth Observation Satellites

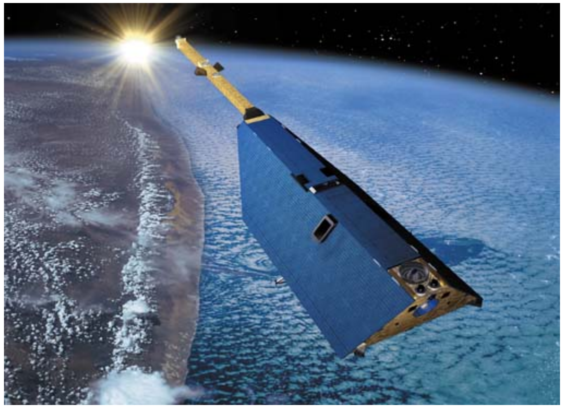


Precise Tracking Data in Near Earth Space



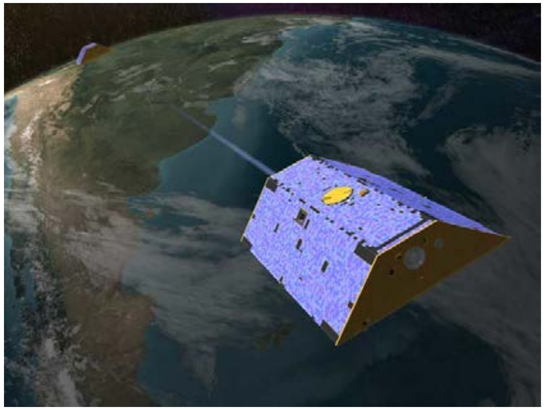
Low Earth Orbiters (LEOs)

CHAMP



CHallenging **M**inisatellite **P**ayload

GRACE



Gravity **R**ecovery **A**nd **C**limate **E**xperiment

GOCE

Gravity and steady-state **O**cean **C**irculation **E**xplorer

Of course, there are many more missions equipped with GPS receivers

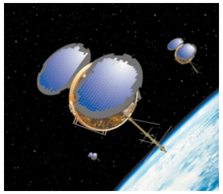
Jason

Jason-2

MetOp-A

Icesat

COSMIC



LEO Constellations

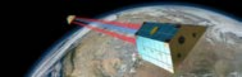
TanDEM-X

Swarm

Sentinel

GRACE-FO

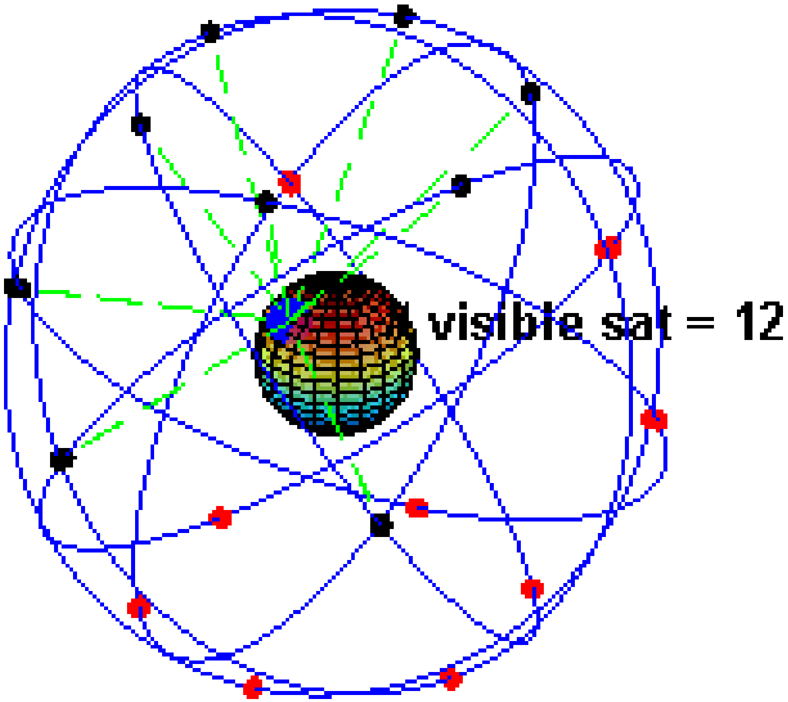
COSMIC-2



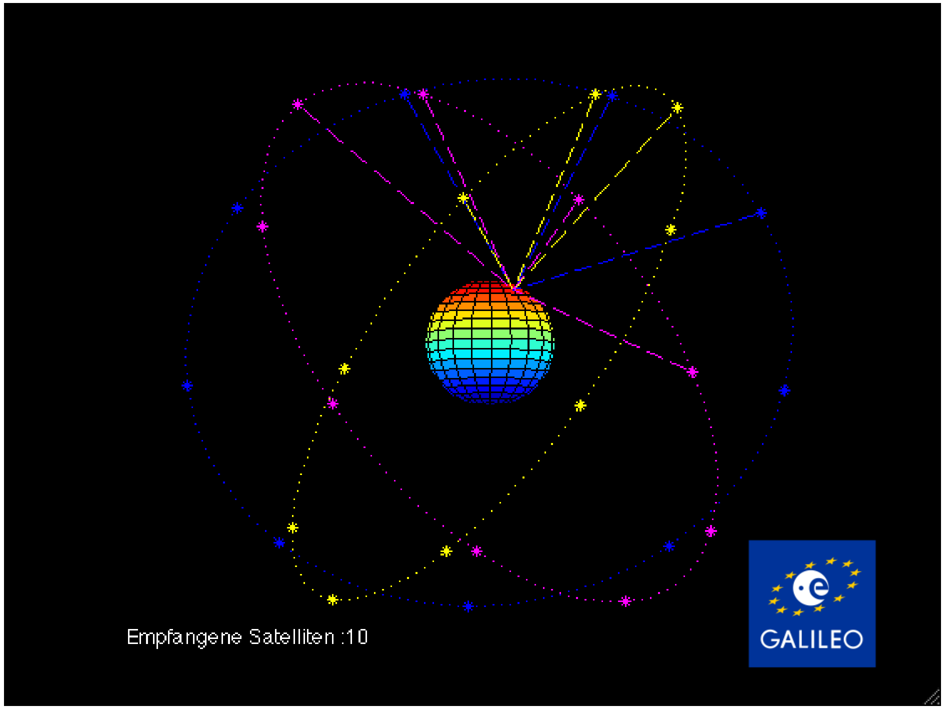
Global Positioning System

Introduction to GPS

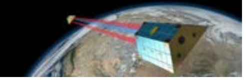
GPS



Galileo



Other Global Navigation Satellite Systems (GNSS) are also available (GLONASS, Galileo, Beidou), but for a long time no multi-GNSS spaceborne receivers were in orbit. This changed with the launches of Fengyun-3, COSMIC-2, Sentinel-6.



Introduction to GPS

GPS: **G**lobal **P**ositioning **S**ystem

Characteristics:

- Satellite system for (real-time) **Positioning** and **Navigation**
- **Global** (everywhere on Earth, up to altitudes of 5000km) and **at any time**
- **Unlimited** number of users
- **Weather-independent** (radio signals are passing through the atmosphere)
- **3-dimensional position**, **velocity** and **time** information

GPS Segments

The GPS consists of **3 main segments**:

- **Space Segment**: the satellites and the constellation of satellites
- **Control Segment**: the ground stations, infrastructure and software for operation and monitoring of the GPS
- **User Segment**: all GPS receivers worldwide and the corresponding processing software

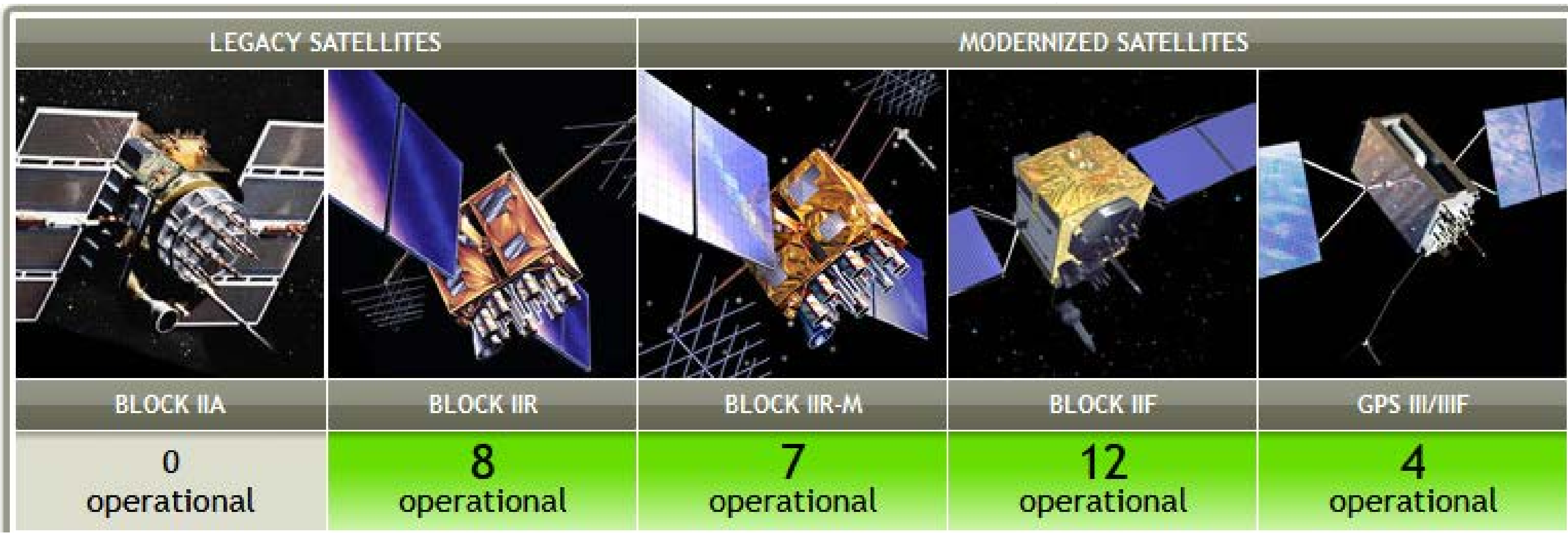
We should add an important **4th segment**:

- **Ground Segment**: all civilian permanent networks of reference sites and the international/regional/local services delivering products for the users

Space Segment

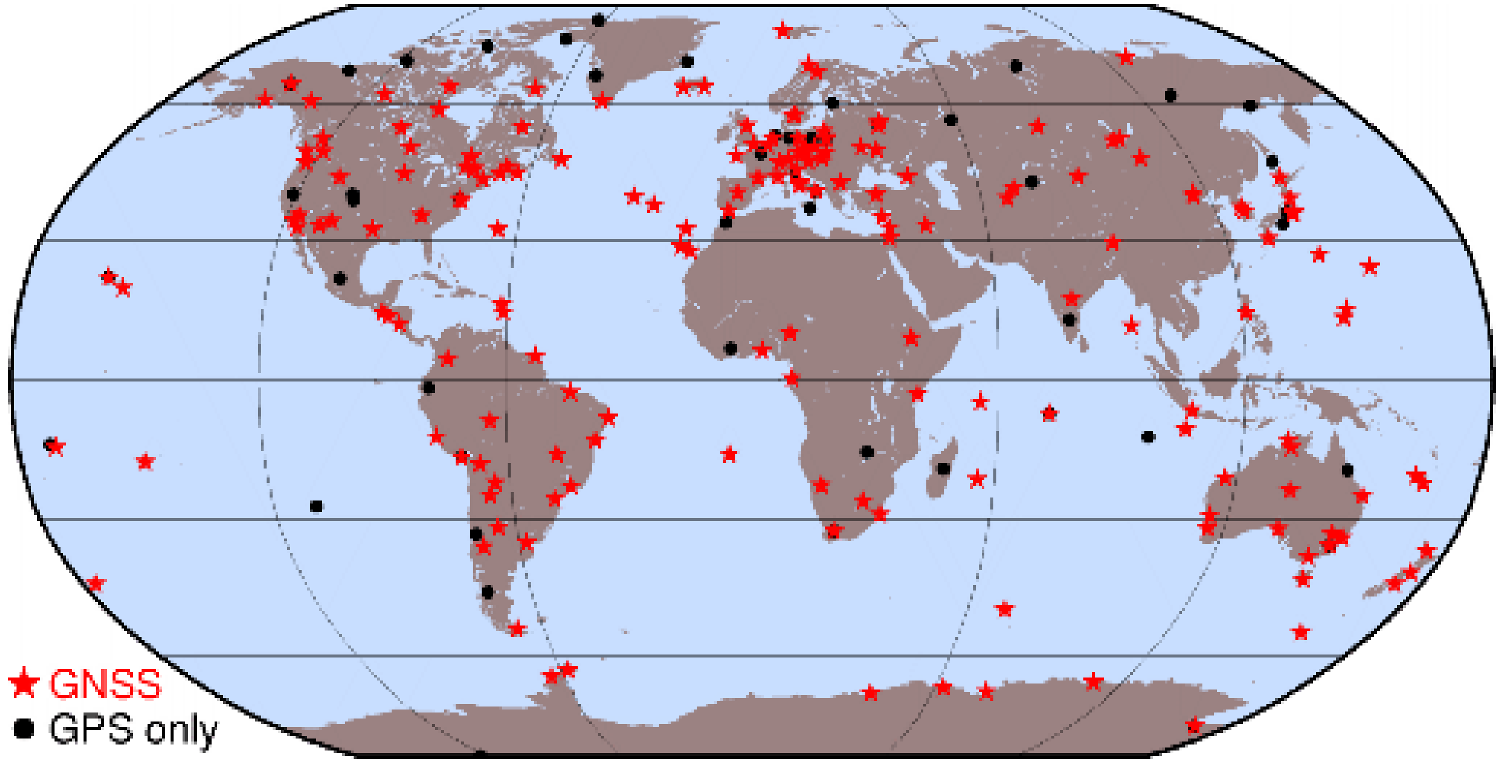
- The space segment nominally consists of **24 satellites**, presently: 31 active GPS satellites

- Constellation design: at least **4 satellites** in view from **any location** on the Earth at **any time**



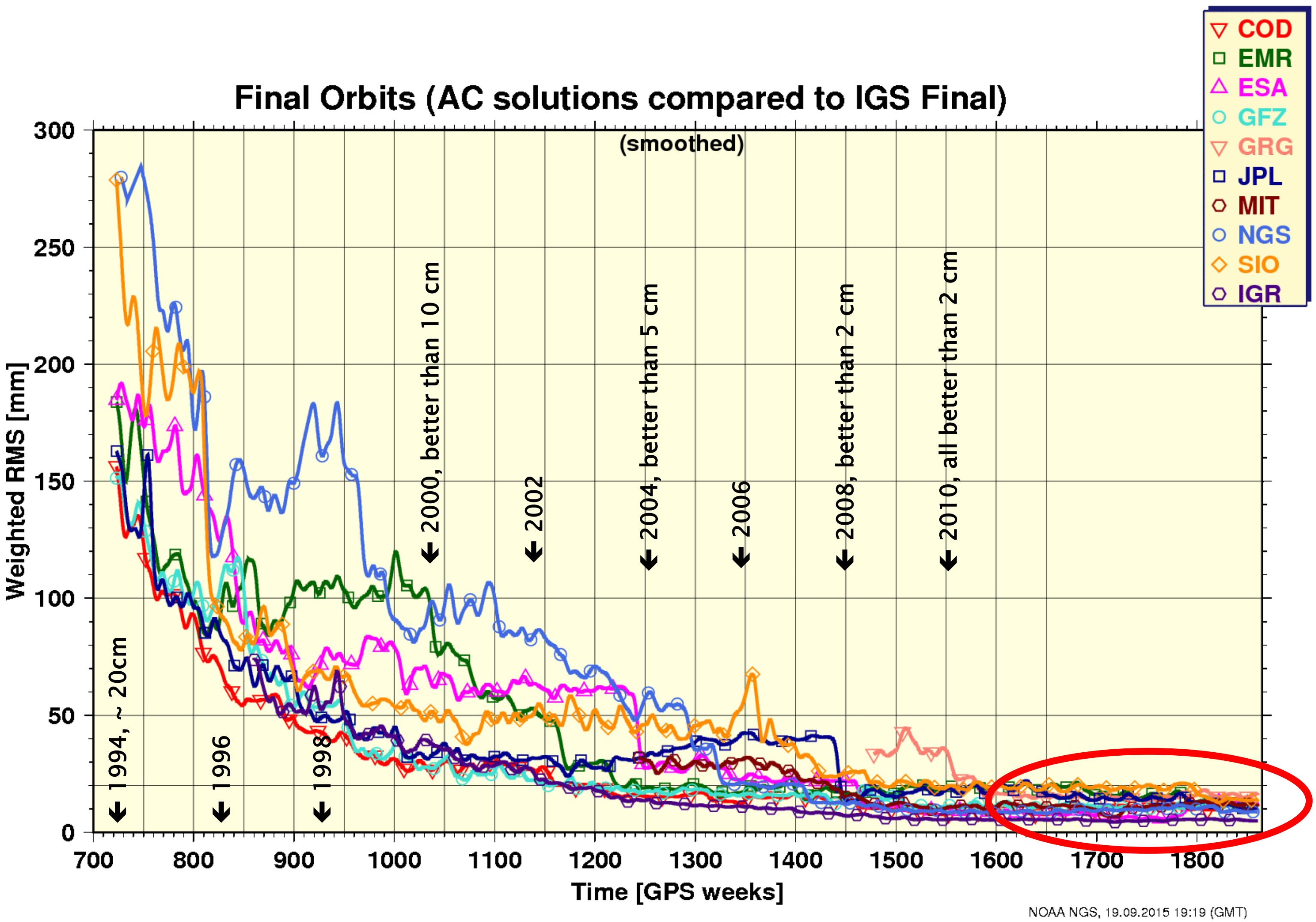
https://www.gps.gov/systems/gps/space/

Global Network of the IGS



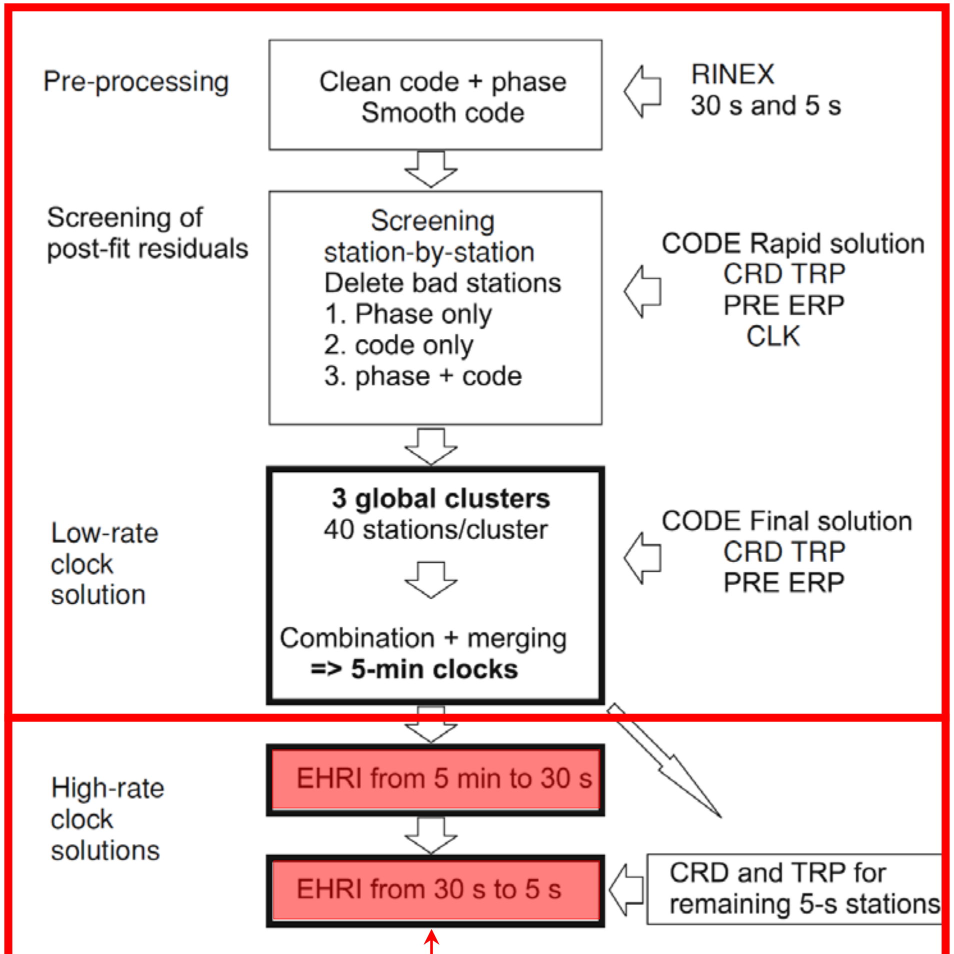
IGS stations used for computation of final orbits at CODE (Dach et al., 2009)

Performance of IGS Final Orbits



http://acc.igs.org/

Computation of Final Clocks at CODE



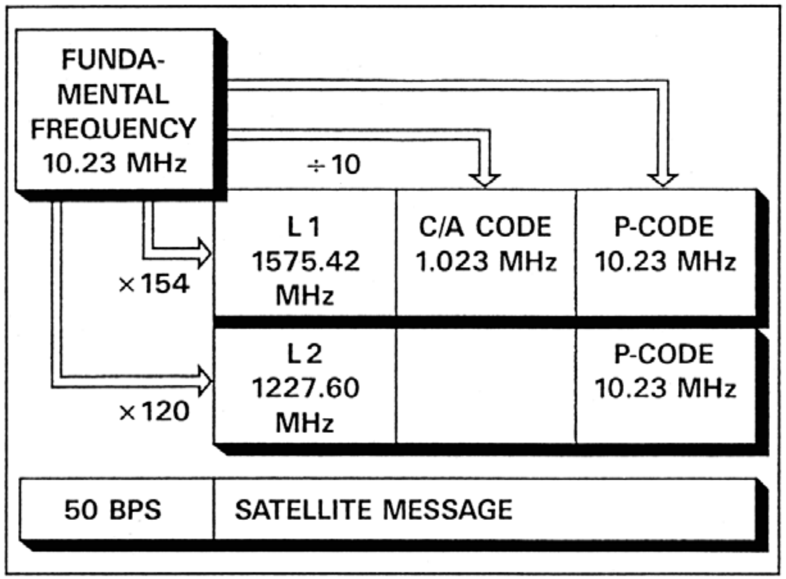
(Bock et al., 2009)

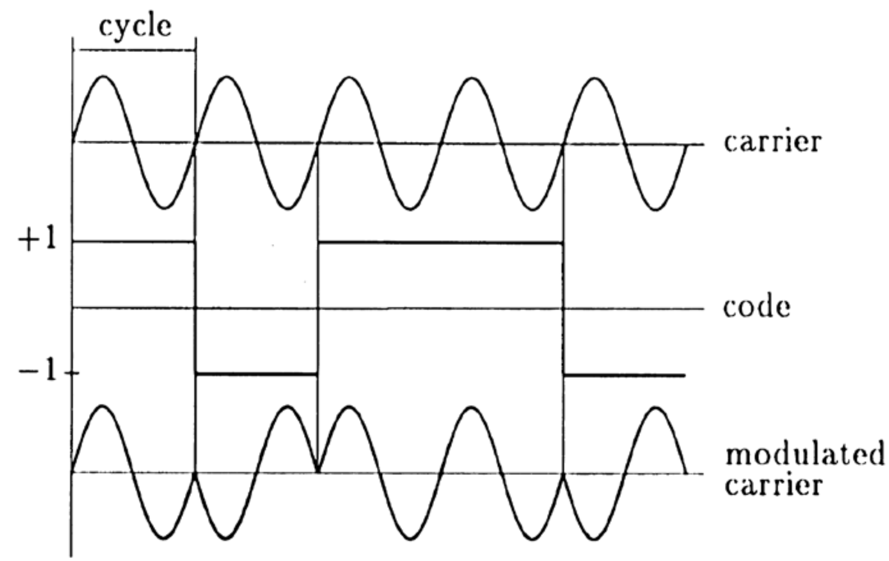
The final clock product with 5 min sampling is based on undifferenced GPS data of typically 120 stations of the IGS network

The IGS 1 Hz network is finally used for clock densification to 5 sec

The 5 sec clocks are interpolated to 1 sec as needed for 1 Hz kinematic LEO POD

GPS Signals





Signals driven by an **atomic clock**

Two **carrier signals** (sine waves):

- **L1**: f = 1575.43 MHz, λ = 19 cm
- **L2**: f = 1227.60 MHz, λ = 24 cm

Bits encoded on carrier by phase modulation:

- **C/A-code** (Clear Access / Coarse Acquisition)
- **P-code** (Protected / Precise)
- **Broadcast/Navigation Message**

Pseudorange / Code Measurements

Code Observations P_i^k are defined as:

$$P_i^k \doteq c\,(T_i - T^k)$$

c Speed of light (in vacuum)

T_i Receiver clock reading at signal reception (in receiver clock time)

T^k GPS satellite clock reading at signal emission (in satellite clock time)

- No actual „**range**“ (distance) because of clock offsets

- **Measurement noise**: ~ 0.5 m for P-code

Code Observation Equation

$$P_i^k = \rho_i^k - c \cdot \Delta t^k + c \cdot \Delta t_i$$

t_i, t^k	GPS time of reception and emission
Δt^k	Satellite clock offset $T^k - t^k$
Δt_i	Receiver clock offset $T_i - t_i$
ρ_i^k	Distance between receiver and satellite $c\,(t_i - t^k)$

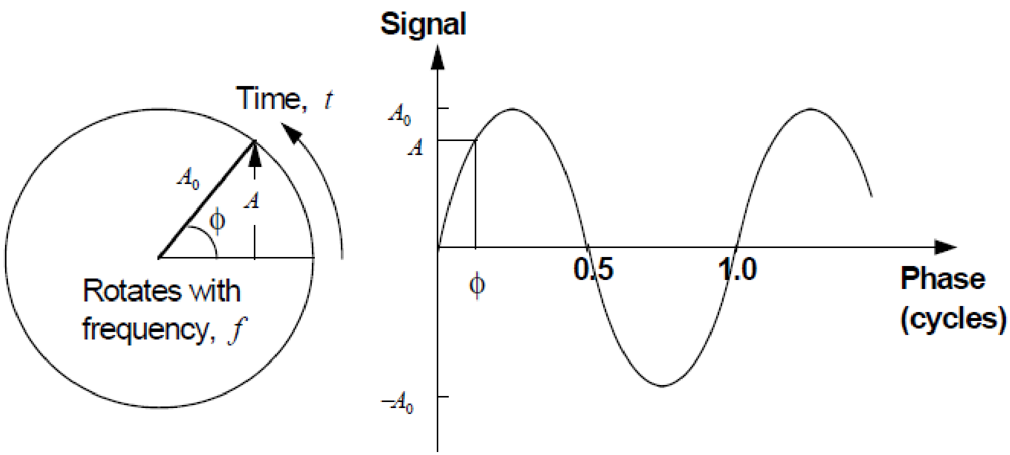
Known from ACs or IGS:

- satellite positions $(x^{k_j}, y^{k_j}, z^{k_j})$
- satellite clock offsets Δt^{k_j}

4 unknown parameters:

- receiver position (x_i, y_i, z_i)
- receiver clock offset Δt_i

Carrier Phase Measurements (1)



Phase ϕ (in cycles) increases linearly with time t :

$$\phi = f \cdot t$$

where f is the frequency

The **satellite** generates with its clock the phase signal ϕ^k. At emission time T^k (in satellite clock time) we have

$$\phi^k = f \cdot T^k$$

The same phase signal, e.g., a wave crest, propagates from the satellite to the receiver, but the receiver measures only the fractional part of the phase and does not know the **integer number of cycles** N_i^k (phase ambiguity):

$$\phi_i^k = \phi^k - N_i^k = f \cdot T^k - N_i^k$$

Carrier Phase Measurements (2)

The **receiver** generates with its clock a **reference phase.** At time of reception T_i of the satellite phase ϕ_i^k (in receiver clock time) we have:

$$\phi_i = f \cdot T_i$$

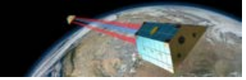
The actual **phase measurement** is the difference between receiver reference phase ϕ_i and satellite phase ϕ_i^k:

$$\psi_i^k = \phi_i - \phi_i^k = f \cdot T_i - (f \cdot T^k - N_i^k) = f \cdot (T_i - T^k) + N_i^k$$

Multiplication with the wavelength $\lambda = c/f$ leads to the **phase observation equation** in meters:

$$\begin{aligned} L_i^k &= \lambda \cdot \psi_i^k = c \cdot (T_i - T^k) + \lambda \cdot N_i^k \\ &= \rho_i^k - c \cdot \Delta t^k + c \cdot \Delta t_i + \lambda \cdot N_i^k \end{aligned}$$

Difference to the pseudorange observation: **integer ambiguity term** N_i^k



Detailed Observation Equation

$$L_i^k = \rho_i^k - c \cdot \Delta t^k + c \cdot \Delta t_i + \cancel{T_i^k} + \cancel{I_i^k} + \lambda \cdot N_i^k$$
$$+ \Delta_{rel} - c \cdot b^k + c \cdot b_i + m_i^k + \epsilon_i^k$$

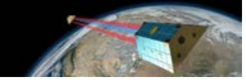
ρ_i^k	Distance between satellite and receiver
Δt^k	Satellite clock offset wrt GPS time
Δt_i	Receiver clock offset wrt GPS time
$\cancel{T_i^k}$	~~Tropospheric delay~~
$\cancel{I_i^k}$	~~Ionospheric delay~~
N_i^k	Phase ambiguity
Δ_{rel}	Relativistic corrections
b^k	Delays in satellite (cables, electronics)
b_i	Delays in receiver and antenna
m_i^k	Multipath, scattering, bending effects
ϵ_i^k	Measurement error

Satellite positions and clocks are known from the IGS

Not existent for LEOs

Cancels out (first order only) when forming the ionosphere-free linear combination:

$$L_c = \frac{f_1^2}{f_1^2 - f_2^2} L_1 - \frac{f_2^2}{f_1^2 - f_2^2} L_2$$



Geometric Distance

Geometric distance ρ_{leo}^{k} is given by:

$$\rho_{leo}^{k} = \left|\boldsymbol{r}_{leo}(t_{leo}) - \boldsymbol{r}^{k}(t_{leo} - \tau_{leo}^{k})\right|$$

$\boldsymbol{r}_{leo}$	Inertial position of LEO antenna phase center at reception time
$\boldsymbol{r}^{k}$	Inertial position of GPS antenna phase center of satellite k at emission time
τ_{leo}^{k}	Signal traveling time between the two phase center positions

Different ways to represent $\boldsymbol{r}_{leo}$:

- **Kinematic** orbit representation
- **Dynamic** or **reduced-dynamic** orbit representation

Different Orbit Representations

Kinematic Orbit Representation (1)

Satellite position $\boldsymbol{r}_{leo}(t_{leo})$ (in inertial frame) is given by:

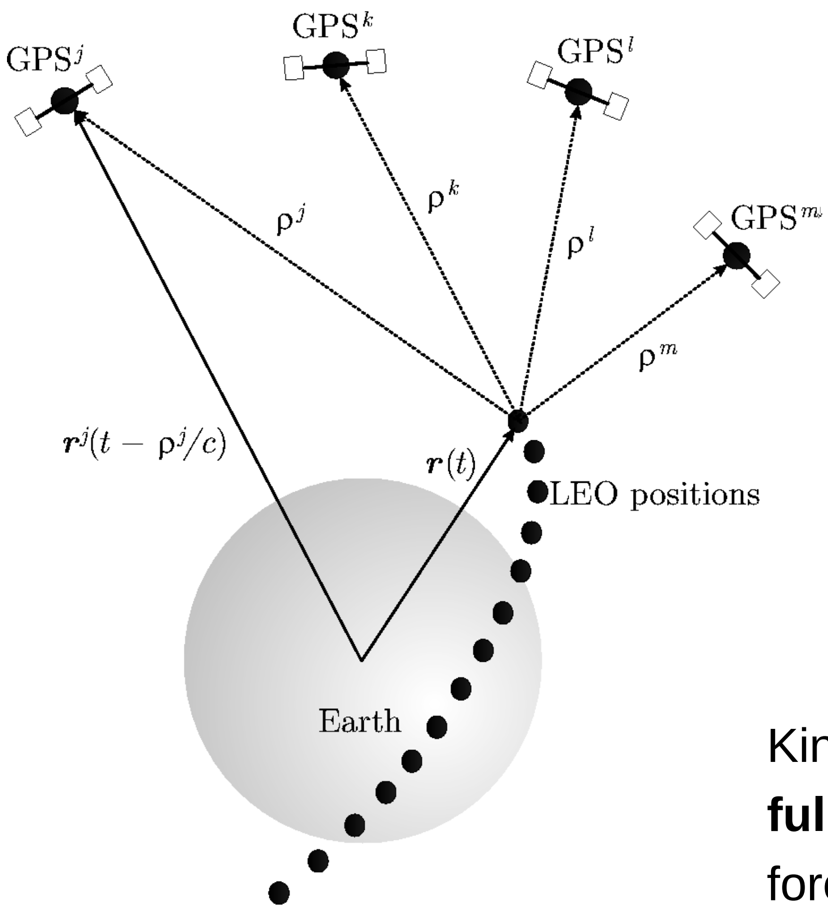
$$\boldsymbol{r}_{leo}(t_{leo}) = \boldsymbol{R}(t_{leo}) \cdot (\boldsymbol{r}_{leo,e,0}(t_{leo}) + \delta\boldsymbol{r}_{leo,e,ant}(t_{leo}))$$

$\boldsymbol{R}$	Transformation matrix from Earth-fixed to inertial frame
$\boldsymbol{r}_{leo,e,0}$	LEO center of mass position in Earth-fixed frame
$\delta\boldsymbol{r}_{leo,e,ant}$	LEO antenna phase center offset in Earth-fixed frame

Kinematic positions $\boldsymbol{r}_{leo,e,0}$ are estimated for each **measurement epoch**:

- Measurement epochs **need not** to be identical with nominal epochs
- Positions are **independent** of models describing the LEO dynamics. Velocities and accelerations cannot be provided in a "strict" sense.

Kinematic Orbit Representation (2)



A kinematic orbit is an ephemeris at **discrete** measurement epochs

Kinematic positions are **fully independent** on the force models used for LEO orbit determination

(Svehla and Rothacher, 2004)

Kinematic Orbit Representation (3)

Measurement epochs
(in GPS time)

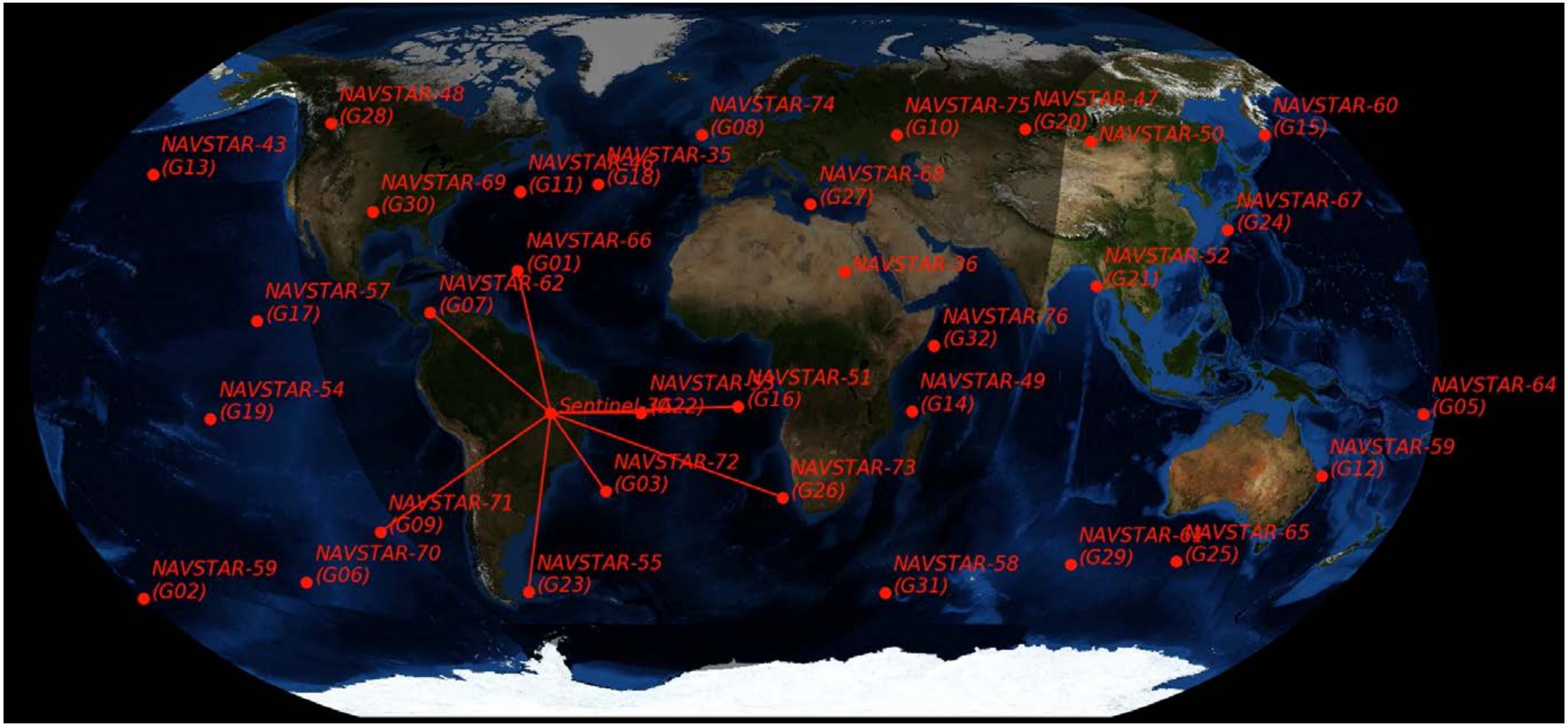
Positions (km)
(Earth-fixed)

```
*  2009 11  2  0  0  0.80678020
PL15  -390.612059  6623.987679   73.104149 193219.797196
*  2009 11  2  0  0  1.80678020
PL15  -389.240315  6624.166512   65.402457 193219.799413
*  2009 11  2  0  0  2.80678020
PL15  -387.868014  6624.336133   57.700679 193219.801634
*  2009 11  2  0  0  3.80678020
PL15  -386.495163  6624.496541   49.998817 193219.803855
*  2009 11  2  0  0  4.80678019
PL15  -385.121760  6624.647724   42.296889 193219.806059
*  2009 11  2  0  0  5.80678019
PL15  -383.747819  6624.789703   34.594896 193219.808280
*  2009 11  2  0  0  6.80678019
PL15  -382.373332  6624.922464   26.892861 193219.810495
*  2009 11  2  0  0  7.80678019
PL15  -380.998306  6625.046003   19.190792 193219.812692
*  2009 11  2  0  0  8.80678019
PL15  -379.622745  6625.160329   11.488692 193219.814899
*  2009 11  2  0  0  9.80678018
PL15  -378.246651  6625.265448    3.786580 193219.817123
```

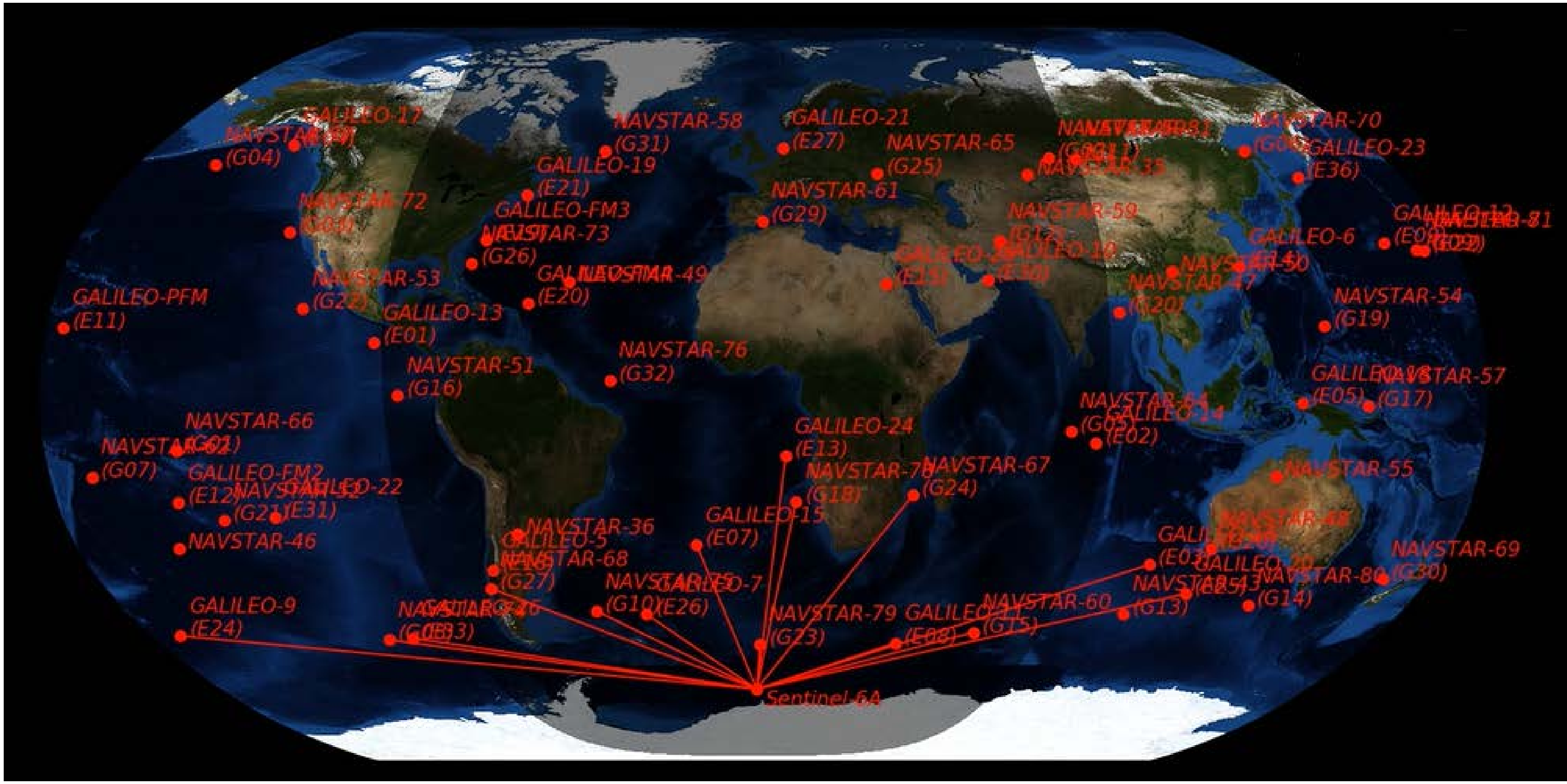
Clock correction to nominal epoch (μs), e.g., to epoch 00:00:03

Excerpt of kinematic GOCE positions at begin of 2 Nov, 2009

Example: Sentinel-3A GPS Tracking



Example: Sentinel-6A multi-GNSS Tracking



Simulated Data Set (1)

The simplified **Code Observation Equation** of the simulation reads as

$$\bar{\rho}_{k,i} = \sqrt{(x_{k,i} - x_{leo,i})^2 + (y_{k,i} - y_{leo,i})^2 + (z_{k,i} - z_{leo,i})^2} + c_{leo,i} \quad , \quad k = 1, \ldots, n_{sat}$$

with

$(x_{k,i}, y_{k,i}, z_{k,i})$ Known inertial position of GPS satellite *k* at epoch *i*

$(x_{leo,i}, y_{leo,i}, z_{leo,i})$ Unknown inertial LEO position at epoch *i*

$c_{leo,i}$ Unknown clock correction of LEO receiver at epoch *i*

The simplified **Phase Observation Equation** of the simulation reads as

$$\bar{\lambda}_{k,i} = \sqrt{(x_{k,i} - x_{leo,i})^2 + (y_{k,i} - y_{leo,i})^2 + (z_{k,i} - z_{leo,i})^2} + c_{leo,i} + b_k \quad , \quad k = 1, \ldots, n_{sat}$$

with the additional parameter

b_k Unknown constant phase bias to the satellite *k*

Simulated Data Set (2)

Tabelle 1: Code observations $\rho'_{k,i}$ contained in the file `OBS_CODE.txt`. They are stored in `template.m` in the array `obs_code(kepo,isat)`. In analogy the phase observations are contained in the file `OBS_PHASE.txt` and are stored in the array `obs_phase(kepo,isat)`. The observation times are also computed in the source code by `t = 10*(kepo-1)`, kepo $= 1, \ldots,$ nepo.

Time t_i (sec)	Sat. 1 (m)	Sat. 2 (m)	Sat. 3 (m)	Sat. 4 (m)	Sat. 5 (m)	etc.
0.0000	19447557.3266	0.0000	0.0000	0.0000	21654965.4010	...
10.0000	19446601.2222	0.0000	0.0000	0.0000	21678374.5718	...
20.0000	19446475.1291	0.0000	0.0000	0.0000	21702249.7103	...
...	...	...	...	...	....	...

Tabelle 2: Positions $(x_{k,i}, y_{k,i}, z_{k,i})$ of the GPS satellites contained in the file `GPS.txt`. They are stored in the array `r_gps(coord,iepo,ksat)`.

x GPS 1 (m)	y GPS 1 (m)	z GPS 1 (m)	x GPS 2	y GPS 2	z GPS 2	etc.
26235000.0000	0.0000	0.0000	-789889.9012	-15196033.7951	-21702185.3748	...
26234971.0435	22468.8157	32088.7944	-751120.9506	-15196458.0361	-21702791.2539	...
26234884.1742	44937.5818	64177.5179	-712350.3922	-15196849.7458	-21703350.6732	...
...	...	...	...	...	....	...

Simulated Data Set (3)

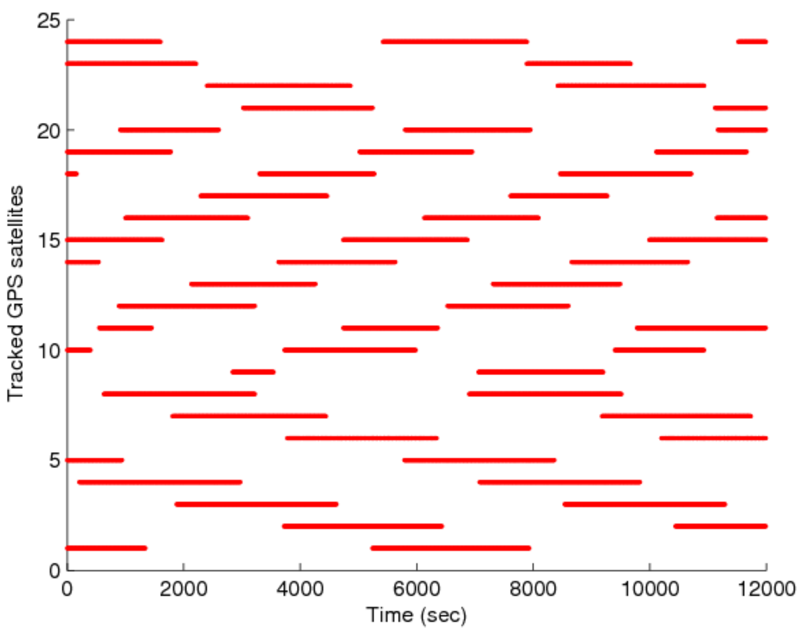
Tabelle 3: Positions $(x_{leo,i}, y_{leo,i}, z_{leo,i})$ and velocities of the LEO satellite contained in the file `LEO.txt`. The positions are stored in the array `r_leo(coord,iepo)`. The velocities are not needed for this project unless the orbit differences shall be plotted in a radial, along-track, cross-track frame instead of the inertial x,y,z frame.

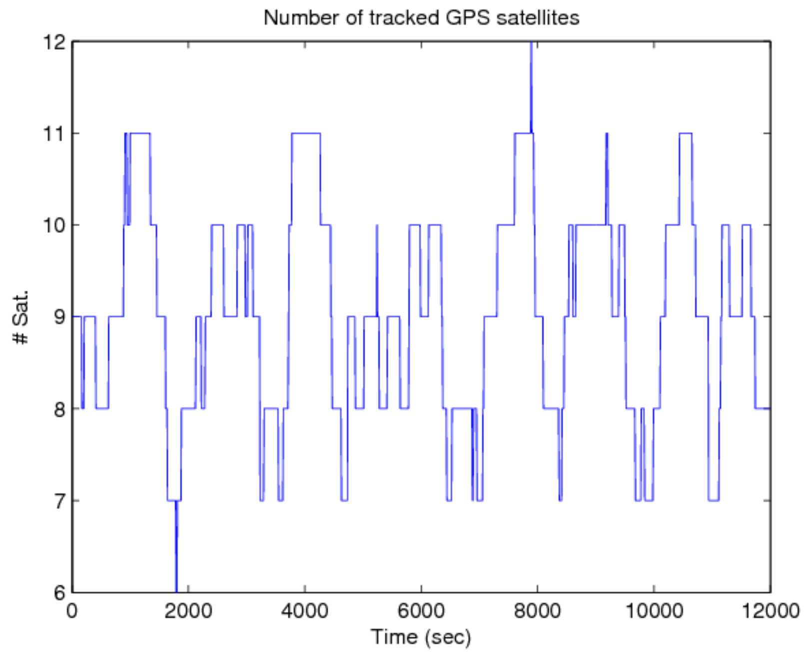
x LEO (m)	y LEO (m)	z LEO (m)	v_x LEO (m/s)	v_y LEO (m/s)	v_z LEO (m/s)
6824717.7284	1203381.8712	0.0000	-0.0000	0.0000	7621.8949
6824309.0437	1203309.8091	76217.4278	-81.7361	-14.4123	7621.4385
6823083.0403	1203093.6316	152425.7274	-163.4621	-28.8228	7620.0693
...	...	...	...	...	...

Tabelle 4: Biases b_k of the phase observations to satellite k contained in the file `BIASES.txt`. They are stored in the array `true_bias(ksat)`.

b_1 (m)	b_2 (m)	b_3 (m)	b_4 (m)	etc.
2319	-1149	2918	-10144	...

Simulated Data Set (4)





Tracking scenario of the simulated data set (left). Up to 12 GPS satellites are at maximum simultaneously visible from the LEO satellite (right). The viewing geometry is continuously changing due to the orbital motion of all satellites.

Dynamic Orbit Representation (1)

Satellite position $\boldsymbol{r}_{leo}(t_{leo})$ (in inertial frame) is given by:

$$\boldsymbol{r}_{leo}(t_{leo}) = \boldsymbol{r}_{leo,0}(t_{leo}; a, e, i, \Omega, \omega, u_0; Q_1, ..., Q_d) + \delta\boldsymbol{r}_{leo,ant}(t_{leo})$$

$\boldsymbol{r}_{leo,0}$	LEO center of mass position
$\delta\boldsymbol{r}_{leo,ant}$	LEO antenna phase center offset
$a, e, i, \Omega, \omega, u_0$	LEO initial osculating orbital elements
$Q_1, ..., Q_d$	LEO dynamical parameters

Satellite trajectory $\boldsymbol{r}_{leo,0}$ is a particular solution of an **equation of motion**

- One set of **initial conditions** (orbital elements) is estimated per arc. Dynamical parameters of the force model may be estimated on request.

Dynamic Orbit Representation (2)

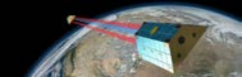
Equation of motion (in inertial frame) is given by:

$$\ddot{\boldsymbol{r}} = -GM\frac{\boldsymbol{r}}{r^3} + \boldsymbol{f}_1(t, \boldsymbol{r}, \dot{\boldsymbol{r}}, Q_1, ..., Q_d)$$

with initial conditions

$$\boldsymbol{r}(t_0) = \boldsymbol{r}(a, e, i, \Omega, \omega, u_0; t_0)$$
$$\dot{\boldsymbol{r}}(t_0) = \dot{\boldsymbol{r}}(a, e, i, \Omega, \omega, u_0; t_0)$$

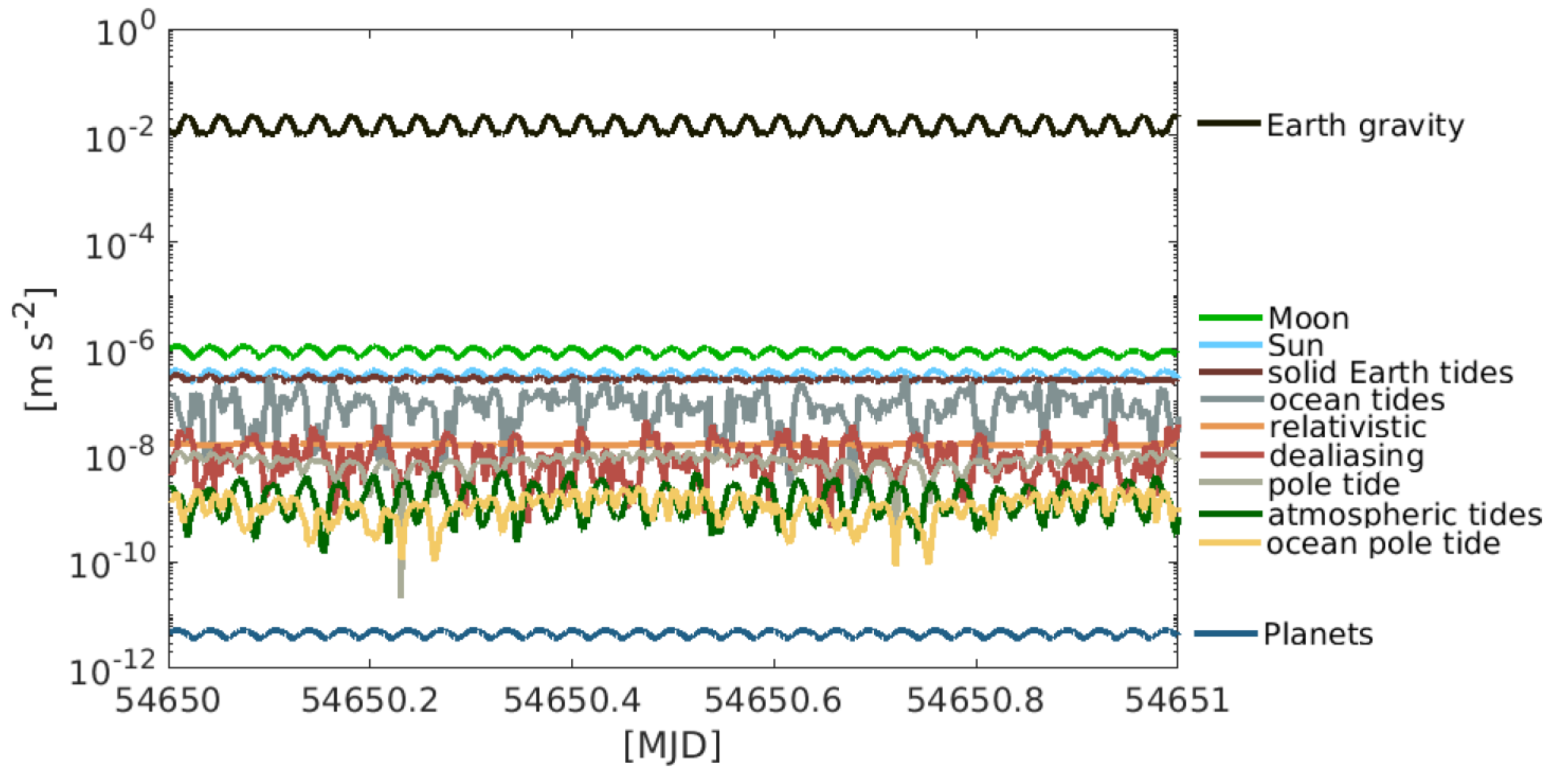
The **acceleration** $\boldsymbol{f}_1$ consists of **gravitational** and **non-gravitational** perturbations taken into account to model the satellite trajectory. Unknown parameters $Q_1, ..., Q_d$ of force models may appear in the equation of motion together with deterministic (known) accelerations given by analytical models.



Perturbing Accelerations of a LEO Satellite

Force	**Acceleration** (m/s²)
Central term of Earth‘s gravity field	8.42
Oblateness of Earth‘s gravity field	0.015
Atmospheric drag	0.00000079
Higher order terms of Earth‘s gravity field	0.00025
Attraction from the Moon	0.0000054
Attraction from the Sun	0.0000005
Direct solar radiation pressure	0.000000097
...	...

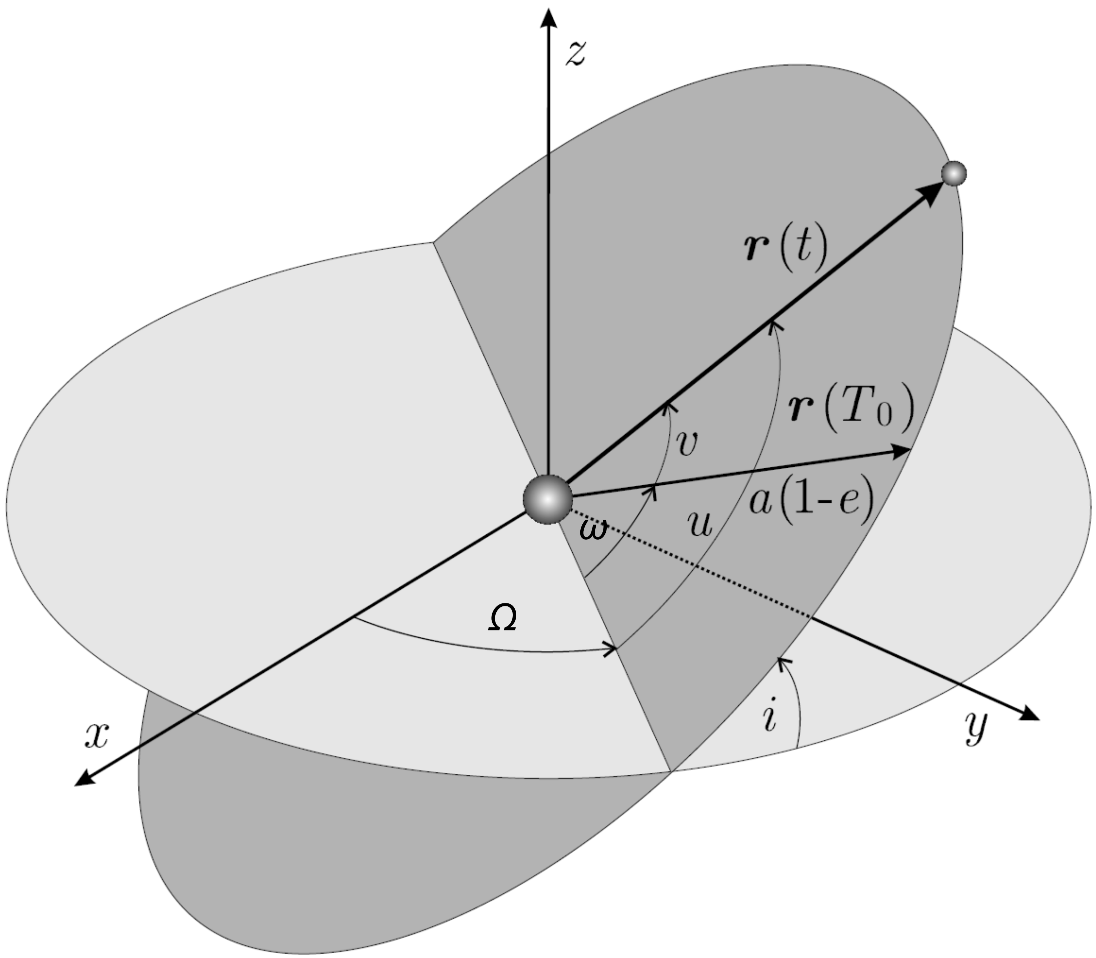
Perturbing Accelerations of a LEO Satellite



Norm of the COST-G benchmark accelerations along a GRACE satellite orbit. The benchmark data set may be used as a reference data set and provides the opportunity to test the implementation of corresponding background models.

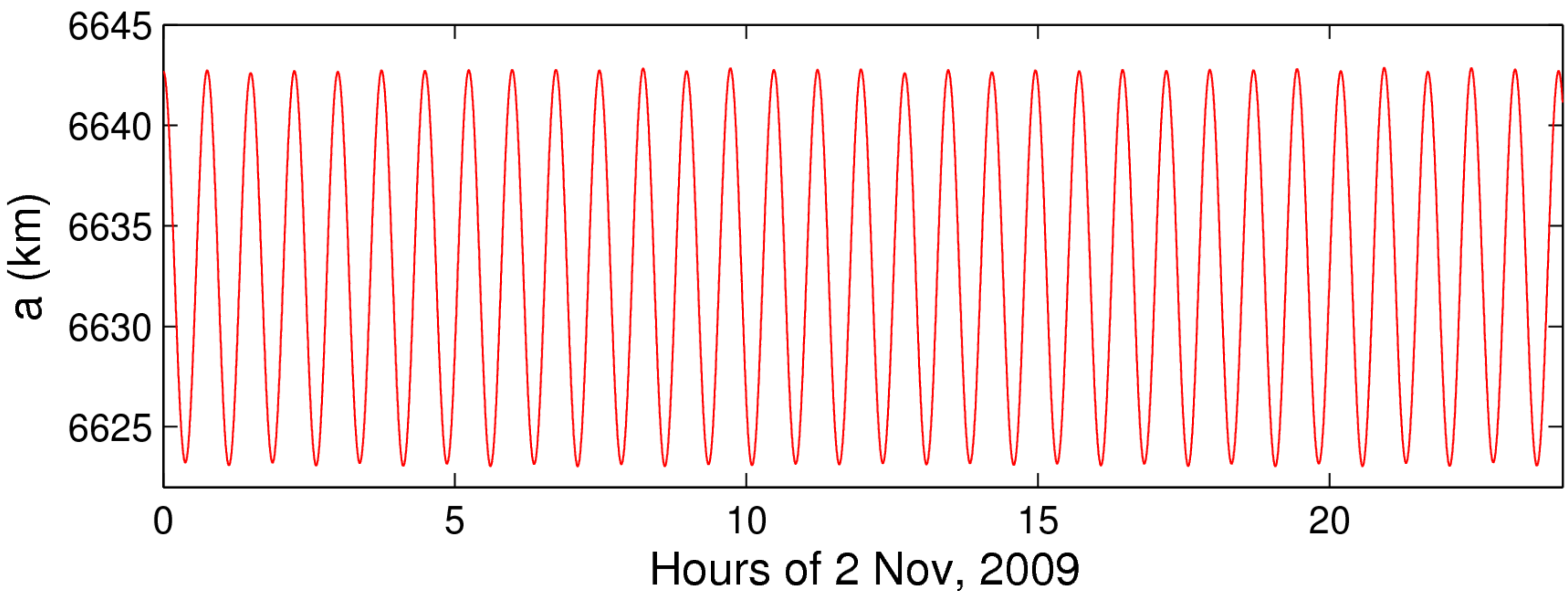
(Mayer-Gürr and Kvas, 2019; Lasser et al., 2020)

Osculating Orbital Elements



(Beutler, 2005)

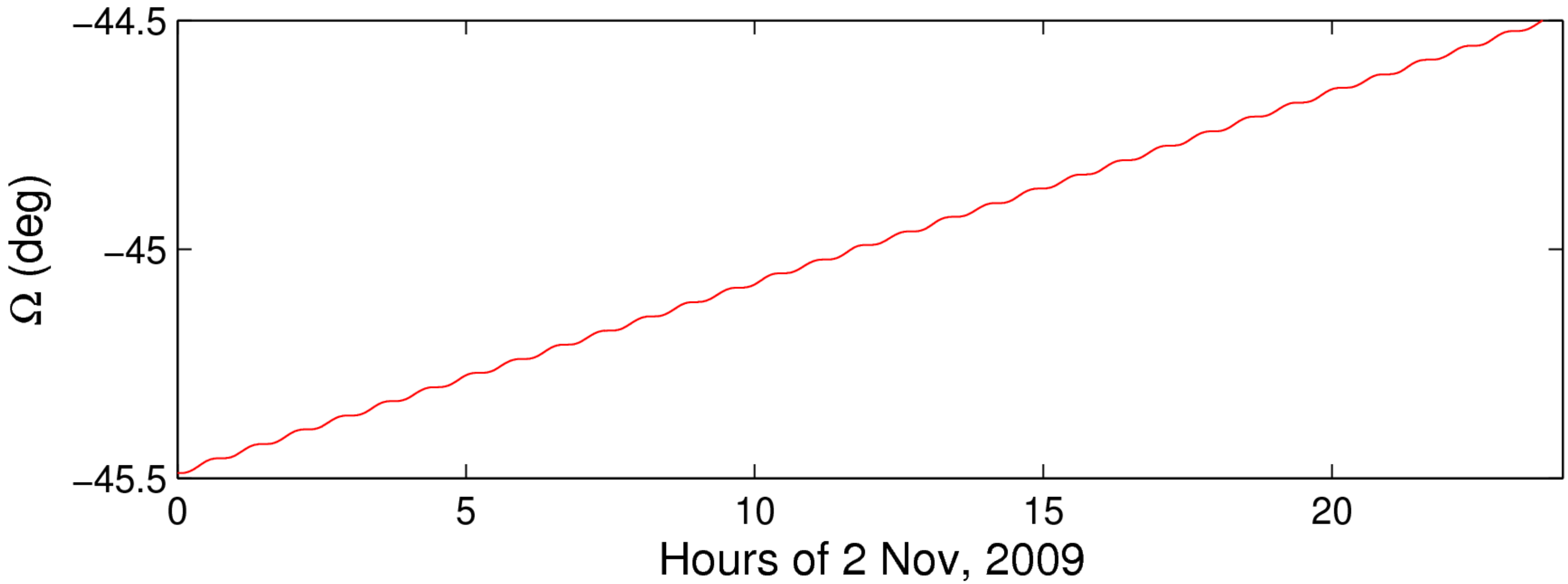
Osculating Orbital Elements of GOCE (1)



Semi-major axis:

Twice-per-revolution variations of about ±10 km around the mean semi-major axis of 6632.9km, which corresponds to a mean altitude of 254.9 km

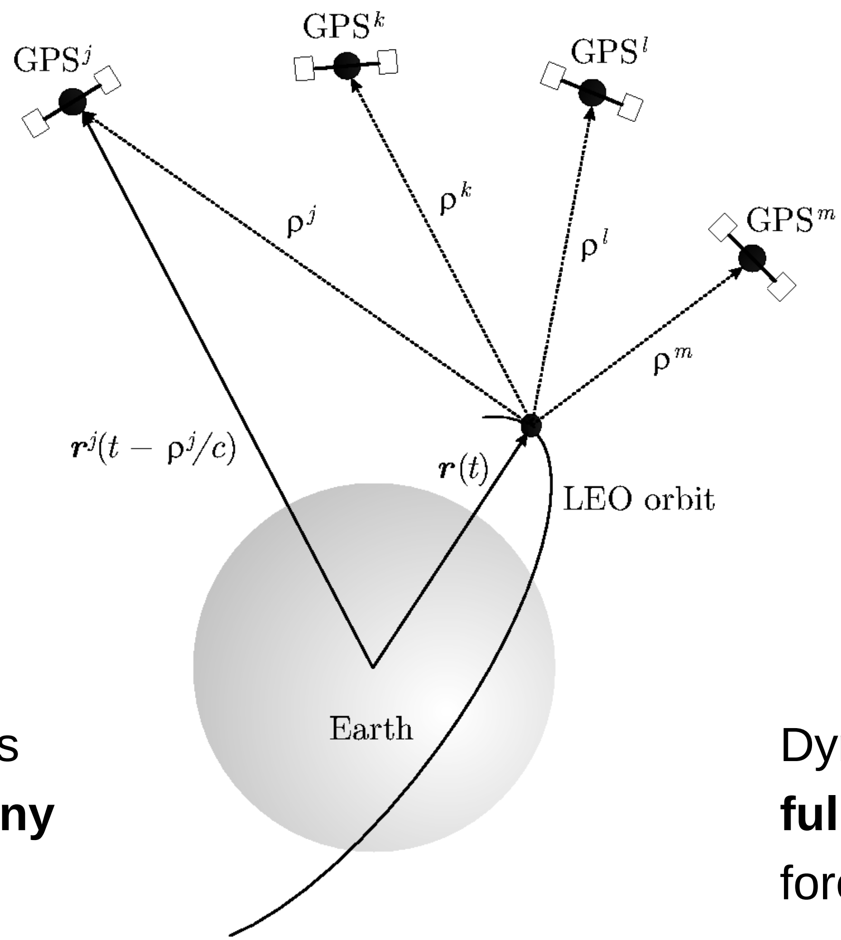
Osculating Orbital Elements of GOCE (2)



Right ascension of ascending node:

Twice-per-revolution variations and linear drift of about +1°/day (360°/365days) due to the sun-synchronous orbit

Dynamic Orbit Representation (3)



Dynamic orbit positions may be computed at **any epoch** within the arc

Dynamic positions are **fully dependent** on the force models used, e.g., on the gravity field model

Reduced-Dynamic Orbit Representation (1)

Equation of motion (in inertial frame) is given by:

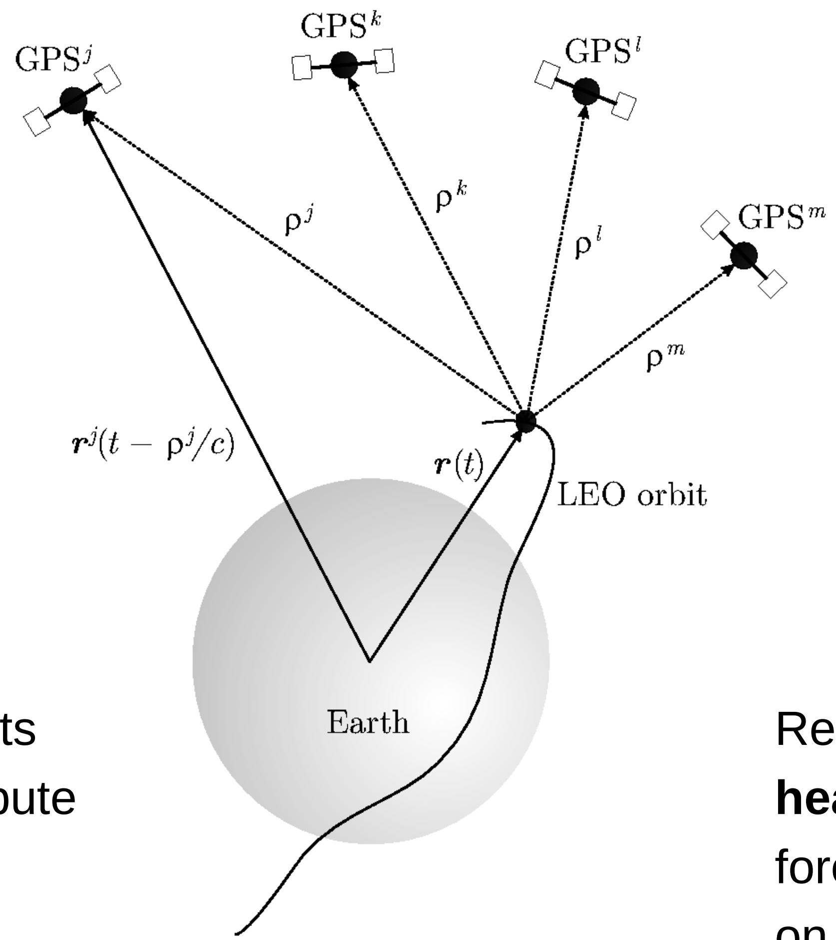
$$\ddot{\boldsymbol{r}} = -GM\frac{\boldsymbol{r}}{r^3} + \boldsymbol{f}_1(t, \boldsymbol{r}, \dot{\boldsymbol{r}}, Q_1, ..., Q_d, P_1, ..., P_s)$$

$P_1, ..., P_s$ Pseudo-stochastic parameters

Pseudo-stochastic parameters are:

- additional empirical parameters characterized by a priori known **statistical properties**, e.g., by expectation values and a priori variances
- useful to **compensate** for deficiencies in dynamic models, e.g., deficiencies in models describing non-gravitational accelerations
- often set up as **piecewise constant accelerations** to ensure that satellite trajectories are continuous and differentiable at any epoch

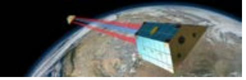
Reduced-Dynamic Orbit Representation (2)



Reduced-dynamic orbits are well suited to compute LEO orbits of **highest quality**

(Jäggi et al., 2006; Jäggi, 2007)

Reduced-dynamic orbits **heavily depend** on the force models used, e.g., on the gravity field model



Reduced-dynamic Orbit Representation (3)

Position epochs
(in GPS time)

Positions (km) &
Velocities (dm/s)
(Earth-fixed)

```
*  2009 11  2  0  0  0.00000000
PL15   -391.718353   6623.836682     79.317661 999999.999999
VL15  13710.157683   1908.731015 -77015.601314 999999.999999
*  2009 11  2  0  0 10.00000000
PL15   -377.980705   6625.284690      2.298385 999999.999999
VL15  13764.602016    987.250587 -77021.193676 999999.999999
*  2009 11  2  0  0 20.00000000
PL15   -364.190222   6625.811136    -74.721213 999999.999999
VL15  13815.825127     65.631014 -77016.232293 999999.999999
*  2009 11  2  0  0 30.00000000
PL15   -350.350131   6625.415949   -151.730567 999999.999999
VL15  13863.820409   -855.995477 -77000.719734 999999.999999
*  2009 11  2  0  0 40.00000000
PL15   -336.463660   6624.099187   -228.719134 999999.999999
VL15  13908.581905  -1777.497047 -76974.660058 999999.999999
*  2009 11  2  0  0 50.00000000
PL15   -322.534047   6621.861041   -305.676371 999999.999999
VL15  13950.104280  -2698.741871 -76938.058807 999999.999999
*  2009 11  2  0  1  0.00000000
PL15   -308.564533   6618.701833   -382.591743 999999.999999
VL15  13988.382807  -3619.598277 -76890.923043 999999.999999
```

Clock corrections
are not provided

Excerpt of reduced-dynamic GOCE positions at begin of 2 Nov, 2009

Principles of Orbit Determination

Principle of Orbit Determination

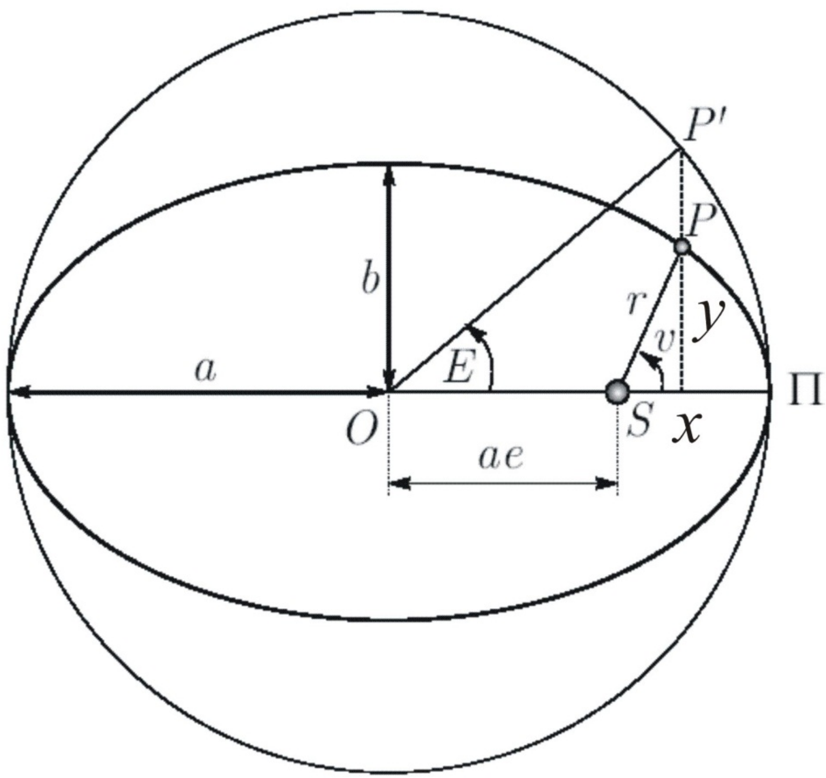
The **actual orbit** $\boldsymbol{r}(t)$ is expressed as a truncated Taylor series:

$$\boldsymbol{r}(t) = \boldsymbol{r}_0(t) + \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}_0}{\partial P_i}(t) \cdot (P_i - P_{0,i})$$

$\boldsymbol{r}_0(t)$	A priori orbit
$\frac{\partial \boldsymbol{r}_0}{\partial P_i}(t)$	Partial derivative of the a priori orbit $\boldsymbol{r}_0(t)$ w.r.t. parameter P_i
$P_{0,i}$	A priori parameter values of the a priori orbit $\boldsymbol{r}_0(t)$
P_i	Parameter values of the improved orbit $\boldsymbol{r}(t)$

A **least-squares** adjustment of spacecraft tracking data yields **corrections** to the a priori parameter values $P_{0,i}$. Using the above equation, the improved (linearized) orbit $\boldsymbol{r}(t)$ may be eventually computed.

A priori orbit generation: Keplerian Orbit



Coordinates in orbital system:

n = mean motion

$n^2\, a^3 = GM$

M = mean anomaly

$M(t) = n\,(t-T_0)$

Kepler's equation:

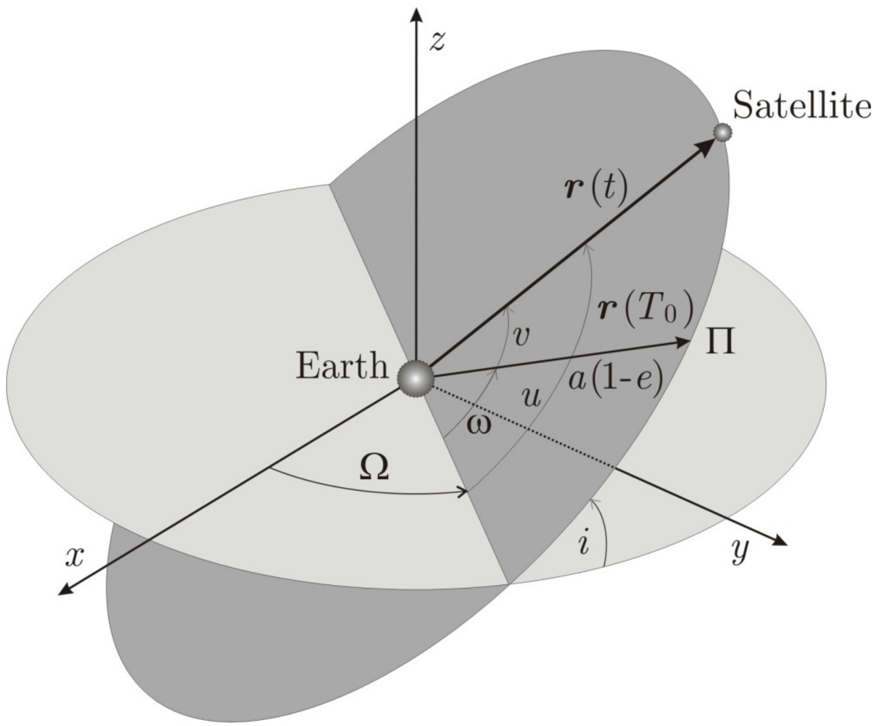
E = eccentric anomaly

$E(t) = M(t) + e \sin E(t)$

$$x = a(\cos E - e)$$

$$y = a\sqrt{1-e^2}\,\sin E$$

A priori orbit generation: Keplerian Orbit



The resulting formulas are those used in the two-body problem for ephemerides calculations. For POD with real data the model is way too simplistic …

Positions in equatorial system:

They follow from the coordinates in the orbital system by adopting three particular rotations:

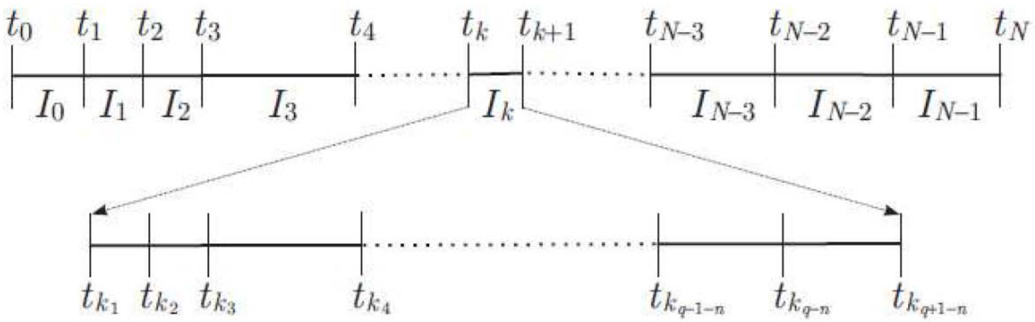
$$\begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} = \mathbf{R}_3(-\Omega)\cdot\mathbf{R}_1(-i)\cdot\mathbf{R}_3(-\omega)\cdot\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

The same holds for the velocities:

$$\begin{pmatrix} \dot{x}_a \\ \dot{y}_a \\ \dot{z}_a \end{pmatrix} = \mathbf{R}_3(-\Omega)\cdot\mathbf{R}_1(-i)\cdot\mathbf{R}_3(-\omega)\cdot\begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix}$$

Numerical Integration (1)

Collocation algorithms (one particular class of numerical integration techniques) are subsequently used to briefly illustrate the principles of numerical integration:



The original intervall is divided into N integration intervals. For each interval I_k a further subdivision is performed according to the order q of the adopted method. At these points t_{k_j} the numerical solution is requested to solve the differential equation system of order n.

(Beutler, 2005)

Numerical Integration (2)

Initial value problem in the interval I_k is given by:

$$\ddot{\mathbf{r}}_k = \mathbf{f}(t, \mathbf{r}_k, \dot{\mathbf{r}}_k)$$

with initial conditions

$$\mathbf{r}_k(t_k) \doteq \mathbf{r}_{k0} \qquad \text{and} \qquad \dot{\mathbf{r}}_k(t_k) \doteq \dot{\mathbf{r}}_{k0}$$

where the initial values are defined as

$$\mathbf{r}_{k0}^{(i)} = \begin{cases} \mathbf{r}_0^{(i)} & ;\, k = 0 \\ \mathbf{r}_{k-1}^{(i)}(t_k) & ;\, k > 0 \end{cases}$$

Numerical Integration (3)

The **collocation method** approximates the solution in the interval I_k by:

$$\mathbf{r}_k(t) \doteq \sum_{l=0}^{q} \frac{1}{l!}(t - t_k)^l \, \mathbf{r}_{k0}^{(l)}$$

The coefficients $\mathbf{r}_{k0}^{(l)}, \, l = 0, \ldots, q$ are obtained by requesting that the numerical solution assumes the initial values and solves the differential equation system at $q - 1$ different epochs $t_{k_j}, \;\; j = 1, \ldots, q - 1$. This leads to the conditions

$$\sum_{l=2}^{q} \frac{(t_{k_j} - t_k)^{l-2}}{(l-2)!} \mathbf{r}_{k0}^{(l)} = \mathbf{f}(t_{k_j}, \mathbf{r}_k(t_{k_j}), \dot{\mathbf{r}}_k(t_{k_j})) \; , \quad j = 1, \ldots, q-1 \, .$$

They are non-linear but can be solved efficiently by an iterative procedure.

(Beutler, 2005)

Partial Derivatives

The partial of the r -th observation w.r.t. orbit parameter P_i may be expressed as

$$\frac{\partial F_r(\boldsymbol{X})}{\partial P_i} = \left(\boldsymbol{\nabla}\left(F_r(\boldsymbol{X})\right)\right)^T \cdot \ \frac{\partial \boldsymbol{r}_0}{\partial P_i}(t)$$

with the gradient given by

$$\left(\boldsymbol{\nabla}\left(F_r(\boldsymbol{X})\right)\right)^T = \left(\frac{\partial F_r(\boldsymbol{X})}{\partial r_{0,1}} \ \frac{\partial F_r(\boldsymbol{X})}{\partial r_{0,2}} \ \frac{\partial F_r(\boldsymbol{X})}{\partial r_{0,3}}\right)$$

if the observations only depend on the geocentric position vector and are referring to only one epoch. The gradient only depends on the type of observations used, whereas the second term is independent of the observation type and related to the **variational equations**. This separates the observation-specific (**geometric**) part from the **dynamic** part.

Variational Equations (1)

For each orbit parameter P_i the corresponding **variational equation** reads as

$$\ddot{\boldsymbol{z}}_{P_i} = \boldsymbol{A}_0 \cdot \boldsymbol{z}_{P_i} + \boldsymbol{A}_1 \cdot \dot{\boldsymbol{z}}_{P_i} + \frac{\partial \boldsymbol{f}_1}{\partial P_i}$$

with $\boldsymbol{z}_{P_i}(t) \doteq \dfrac{\partial \boldsymbol{r}_0}{\partial P_i}(t)$ and the 3 x 3 matrices defined by

$$A_{0[i;k]} \doteq \frac{\partial f_i}{\partial r_{0,k}} \qquad \text{and} \qquad A_{1[i;k]} \doteq \frac{\partial f_i}{\partial \dot{r}_{0,k}}$$

f_i i -th component of the total acceleration $\boldsymbol{f}$

$r_{0,k}$ k-th component of the geocentric position $\boldsymbol{r}_0$

For each orbit parameter P_i the **variational equation** is a linear differential equation system of second order in time. Their solutions are all needed for orbit determination.

Variational Equations (2)

The variational equation is a linear, **homogeneous** system with initial values

$$\boldsymbol{z}_{P_i}(t_0) \neq \boldsymbol{0} \quad \text{and} \quad \dot{\boldsymbol{z}}_{P_i}(t_0) \neq \boldsymbol{0} \quad \text{for} \quad P_i \in \{a, e, i, \Omega, \omega, u_0\}$$

and a linear, **inhomogeneous** system with initial values

$$\boldsymbol{z}_{P_i}(t_0) = \boldsymbol{0} \quad \text{and} \quad \dot{\boldsymbol{z}}_{P_i}(t_0) = \boldsymbol{0} \quad \text{for} \quad P_i \in \{Q_1, ..., Q_d\}$$

Let us assume that the functions $\boldsymbol{z}_{O_j}(t), j = 1, ..., 6$ are the partials w.r.t. the six parameters $O_j, j = 1, ..., 6$ defining the initial conditions at time t_0. The ensemble of these six functions forms one **complete system** of solutions of the homogeneous part of the variational equation, which allows to obtain the solution of the inhomogeneous system by the method of "**variation of constants**".

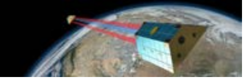
Variational Equations (3)

Note that the solutions $\boldsymbol{z}_{P_i}(t)$ of the variational equation and its time derivative may be expressed with the **same** functions $\alpha_{O_j P_i}(t)$ as a linear combination with the homogeneous solutions $\boldsymbol{z}_{O_j}(t)$ and $\dot{\boldsymbol{z}}_{O_j}(t)$, respectively.Therefore, only the six initial value problems associated with the initial conditions have to be actually treated as differential equation systems. Their solutions have to be either obtained approximately, or by numerical integration techniques.

All variational equations related to dynamical orbit parameters may be reduced to **definite integrals**. They can be efficiently solved numerically, e.g., by a Gaussian quadrature technique.

It must be emphasized that each additional orbit parameter requires an additional numerical solution of a definite integral. In view of the potentially large number of orbit parameters, it is advantagous that for **pseudo-stochastic orbit parameters** an explicit numerical quadrature of the definite integrals can be avoided.

(Jäggi, 2007)



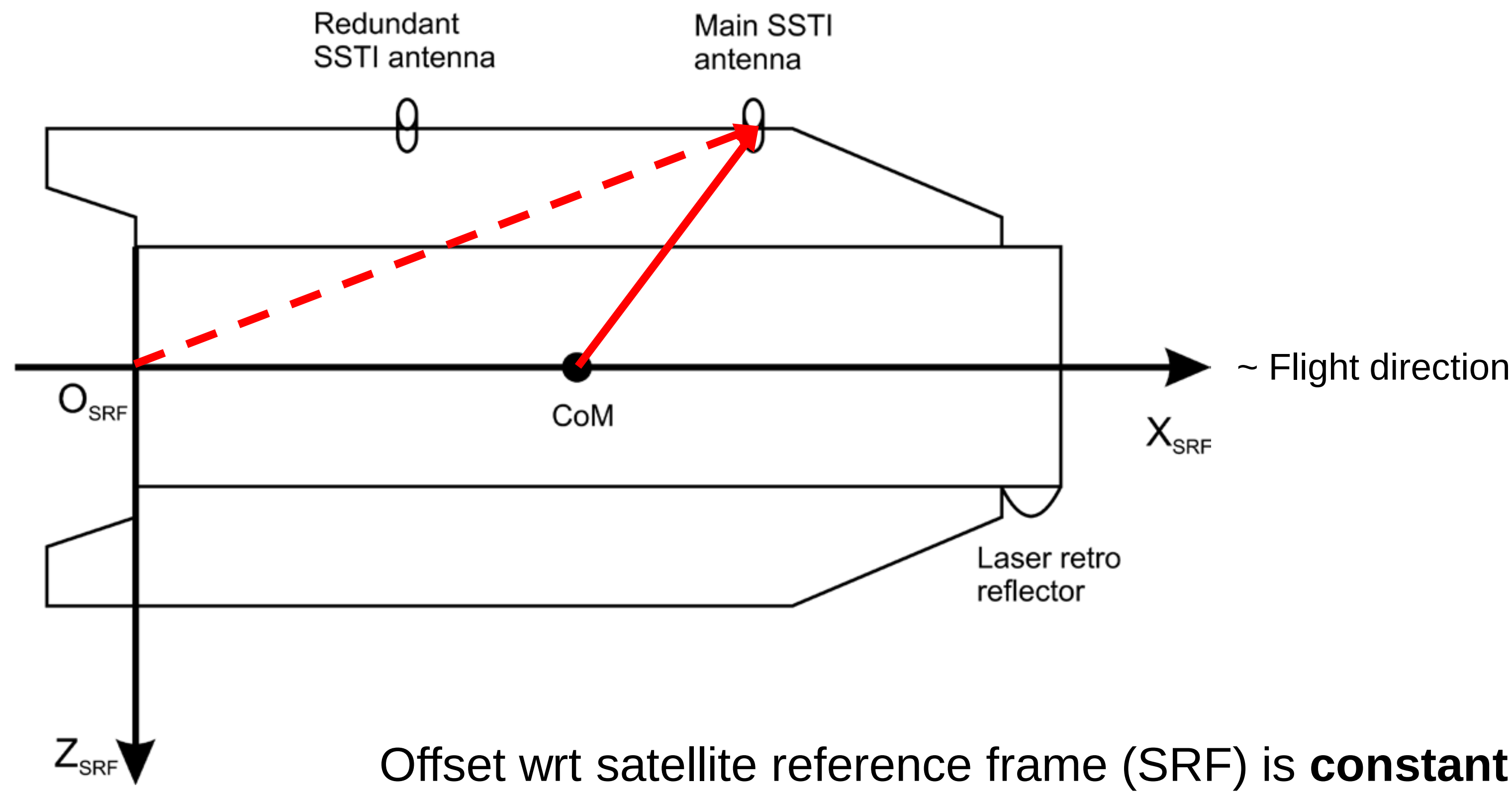
GPS-based LEO POD

LEO Sensor Offsets

Phase center offsets $\delta \boldsymbol{r}_{leo,ant}$:

- are needed in the inertial or Earth-fixed frame and have to be transformed from the satellite frame using **attitude data** from the star-trackers
- consist of a frequency-independent **instrument offset**, e.g., defined by the center of the instrument‘s mounting plane (CMP) in the satellite frame
- consist of frequency-dependent **phase center offsets** (PCOs), e.g., defined wrt the center of the instrument‘s mounting plane in the antenna frame (ARF)
- consist of frequency-dependent **phase center variations** (PCVs) varying with the direction of the incoming signal, e.g., defined wrt the PCOs in the antenna frame

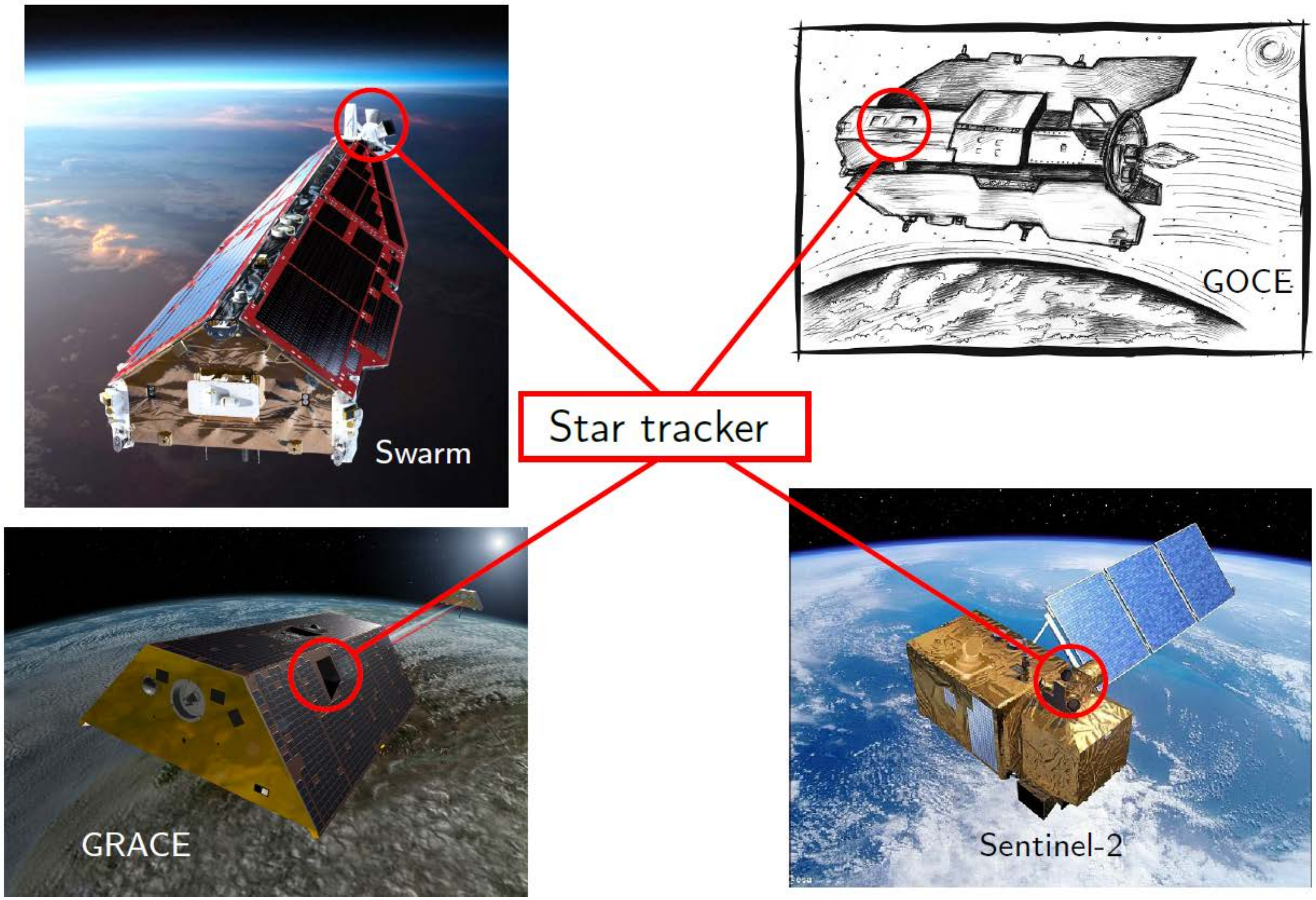
Example: GOCE Sensor Offsets (1)



Offset wrt satellite reference frame (SRF) is **constant**

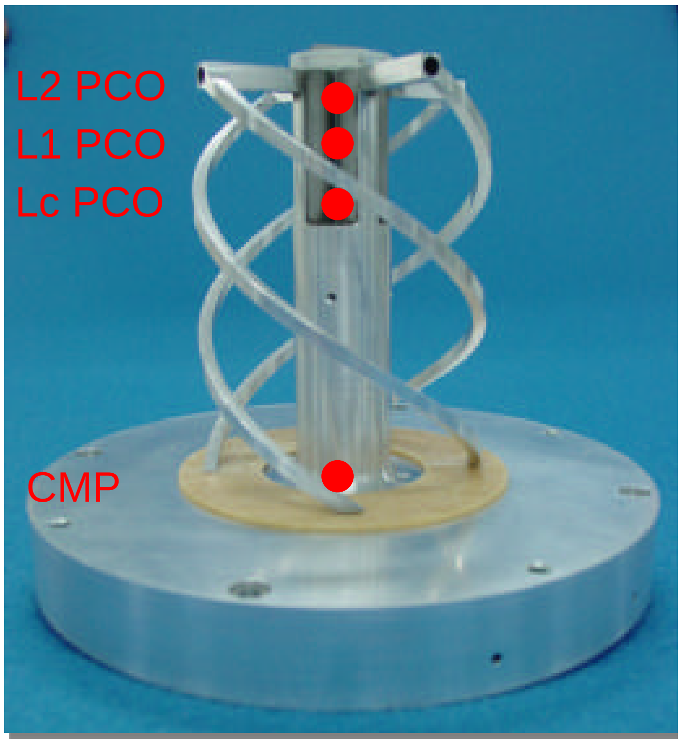
Offset wrt center of mass (CoM) is **slowly varying**

Example: GOCE Sensor Offsets (2)



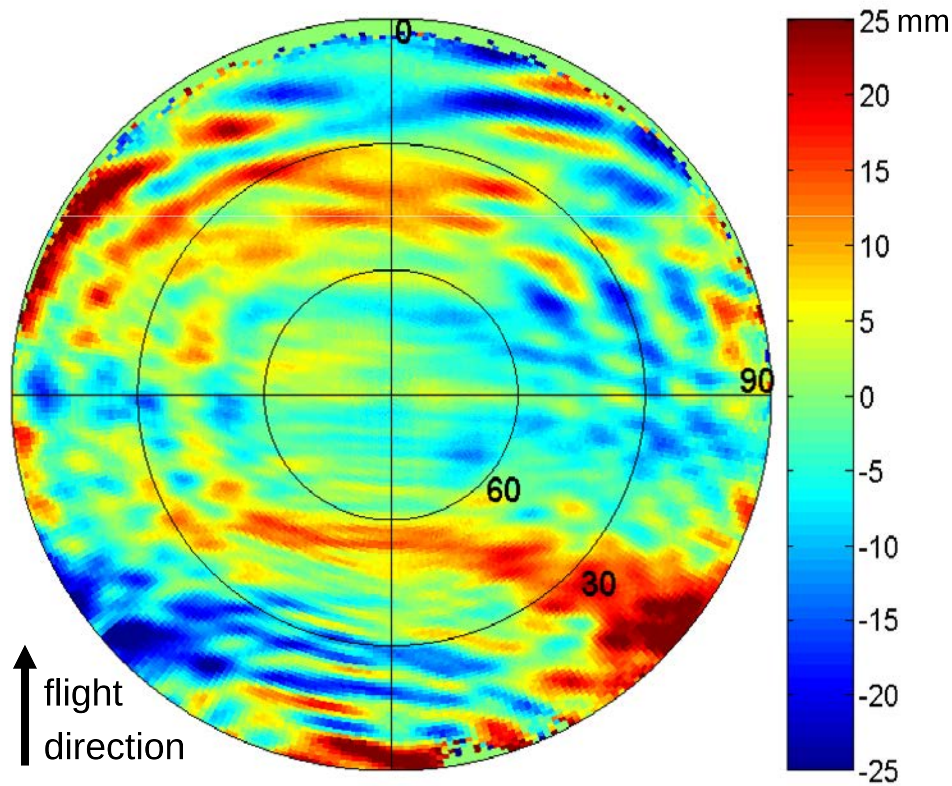
Spaceborne GPS Antennas: GOCE

L1, L2, Lc phase center offsets



Measured from ground calibration
in anechoic chamber

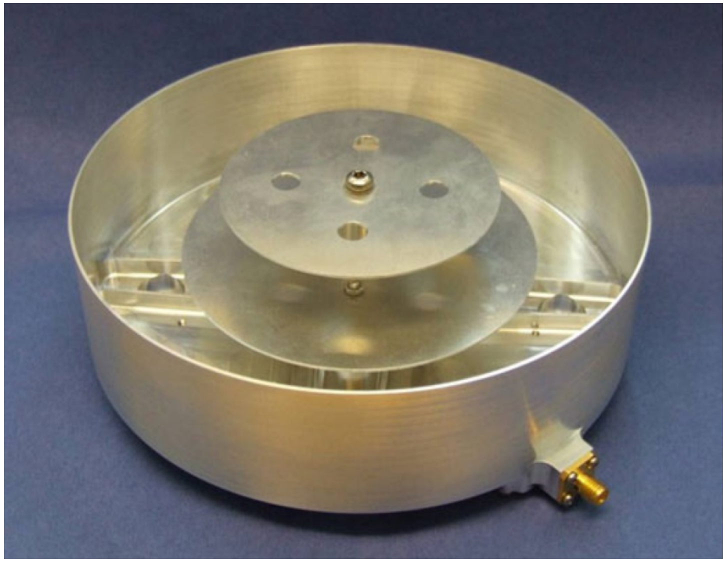
Lc phase center variations



Empirically derived during orbit determination
according to Jäggi et al. (2009)

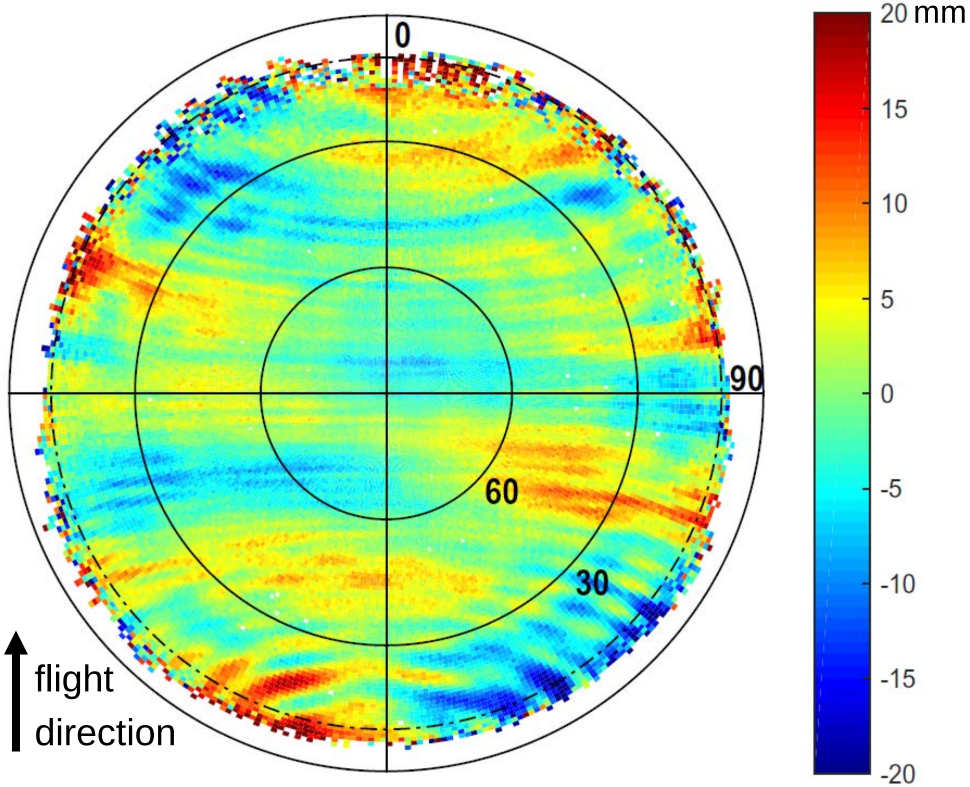
Spaceborne GPS Antennas: Swarm

Swarm GPS antenna



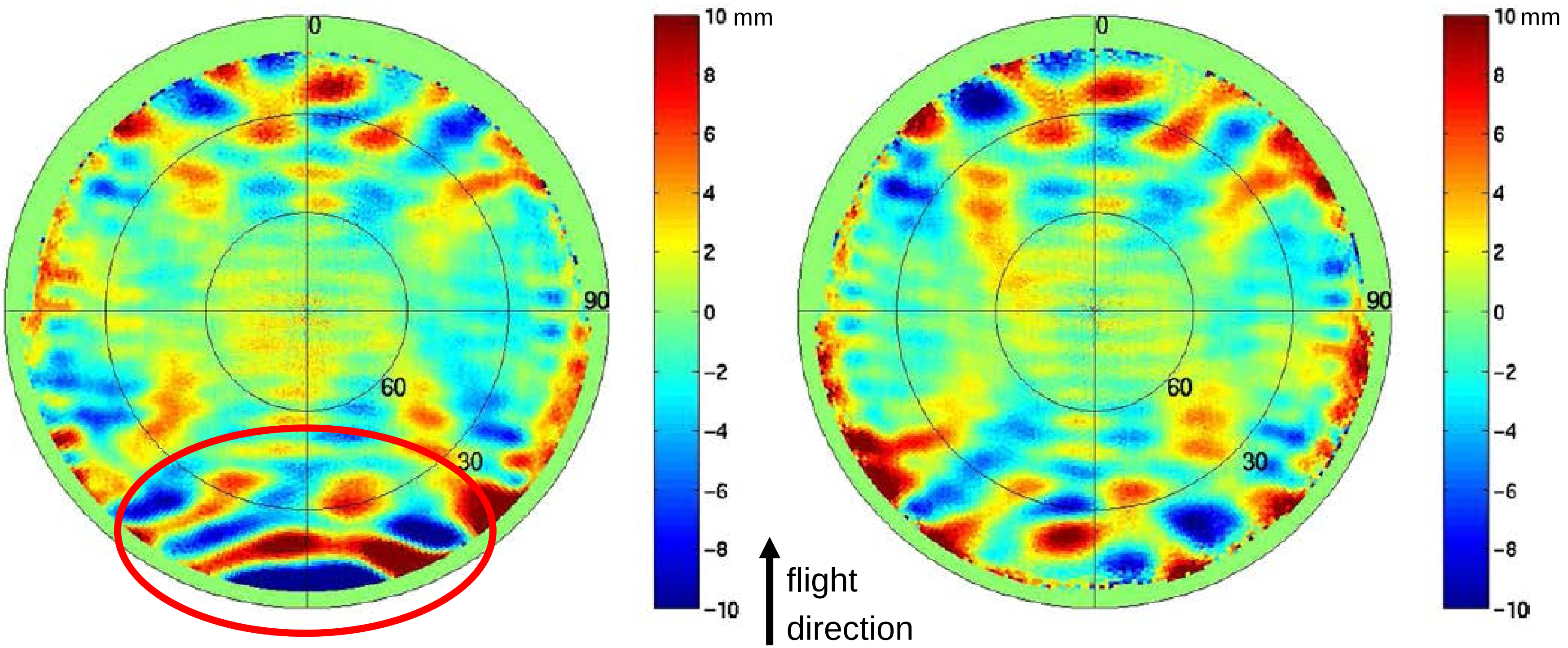
Multipath shall be minimzed by chokering

Lc phase center variations



Empirically derived during orbit determination according to Jäggi et al. (2016)

Spaceborne GPS Antennas: GRACE

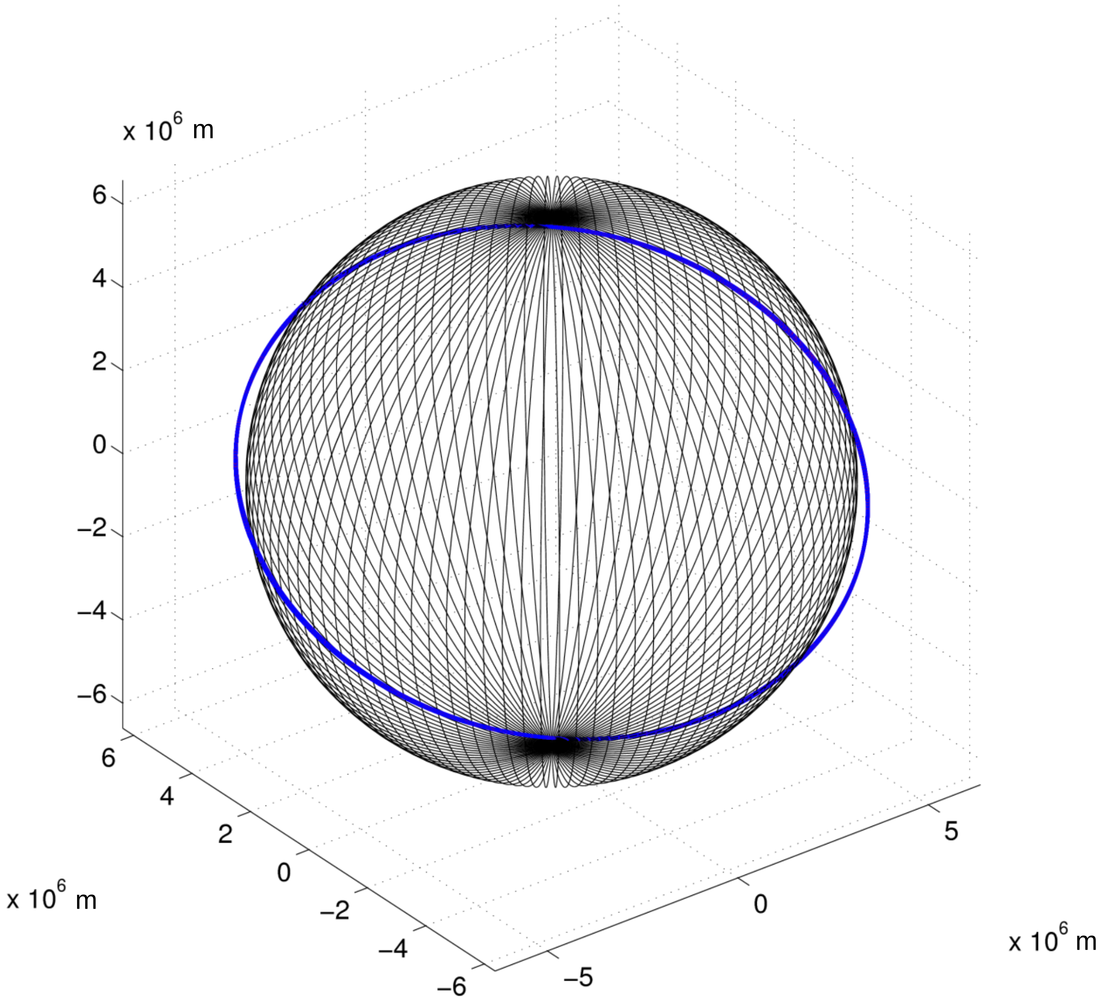


GRACE-A
(occultation antenna switched on)

GRACE-B

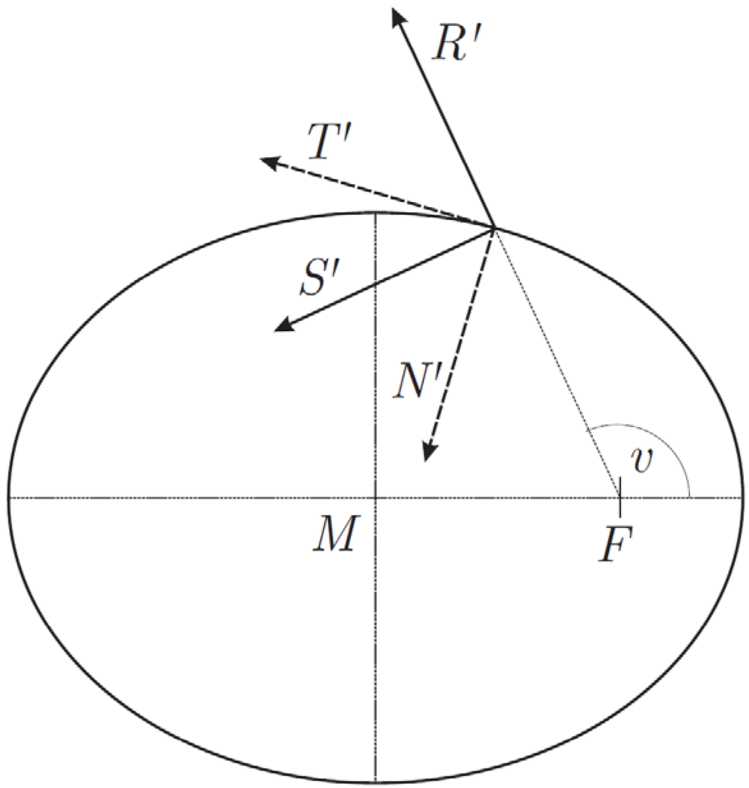
(Jäggi et al., 2009)

Visualization of Orbit Solutions



It is more instructive to look at differences between orbits in well suited coordinate systems …

Co-Rotating Orbital Frames



R, S, C unit vectors are pointing:

- into the radial direction
- normal to **R** in the orbital plane
- normal to the orbital plane (cross-track)

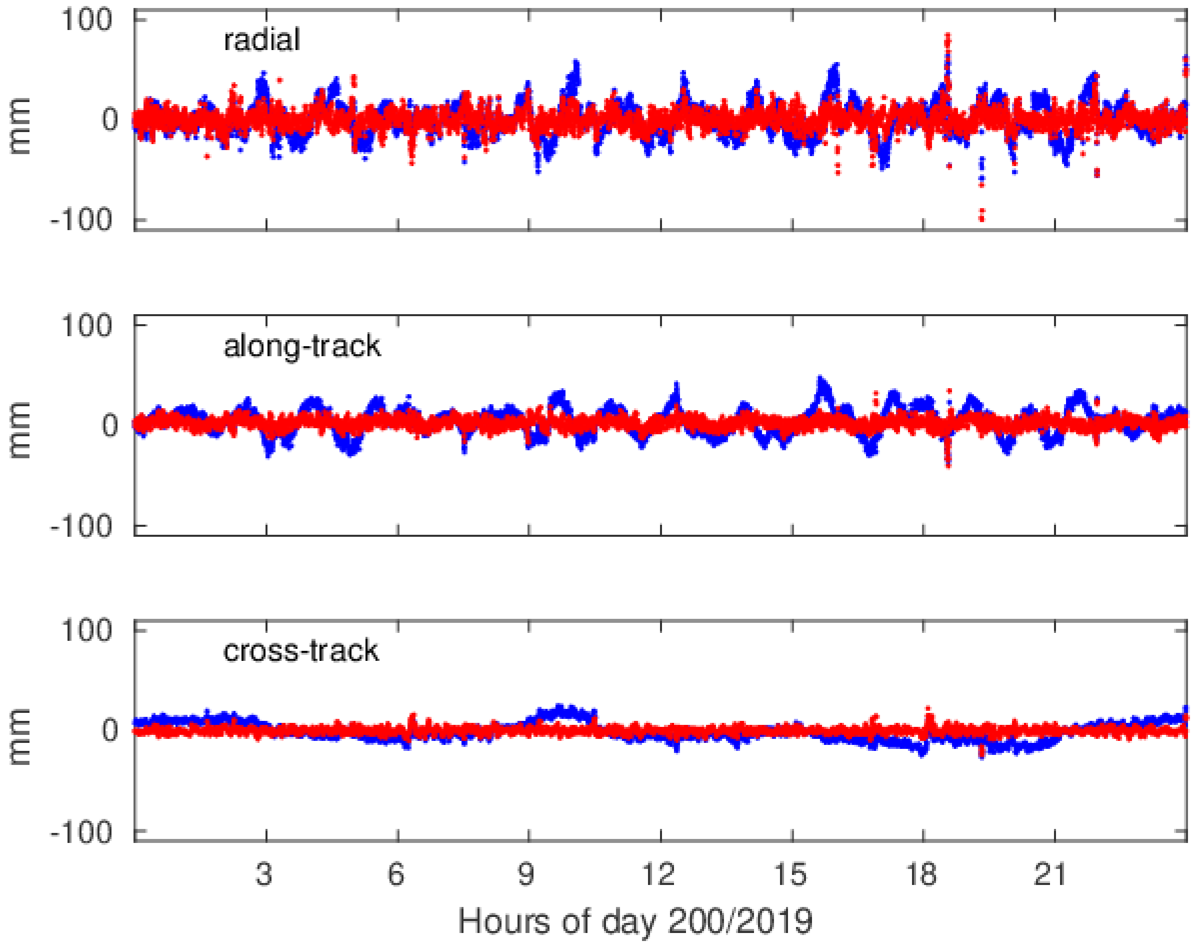
T, N, C unit vectors are pointing:

- into the tangential (along-track) direction
- normal to **T** in the orbital plane
- normal to the orbital plane (cross-track)

Small eccentricities: **S**~**T** (velocity direction)

Orbit Differences KIN-RD (Sentinel-3A)

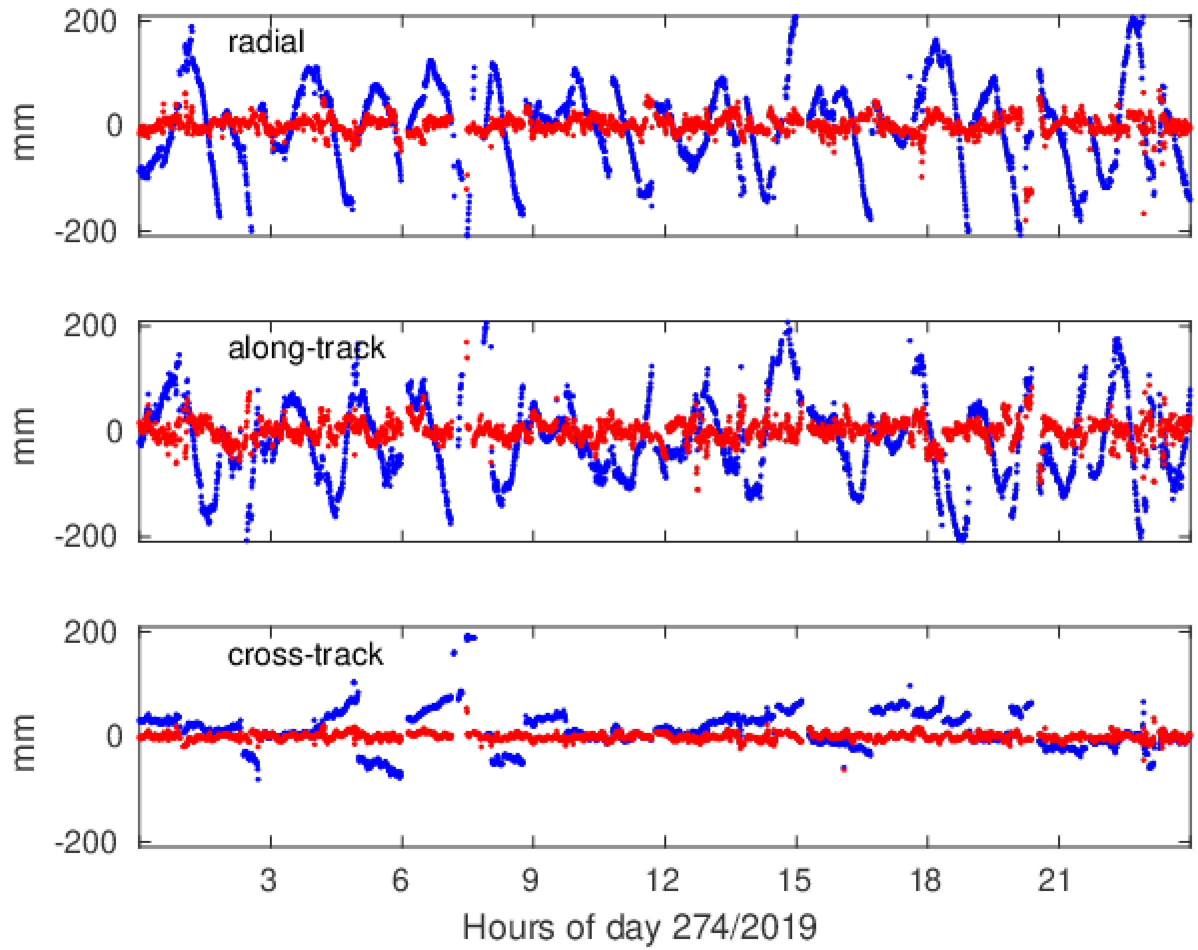
Differences at epochs of kin. positions



Comparison of ambiguity-float solutions and ambiguity-fixed solutions.

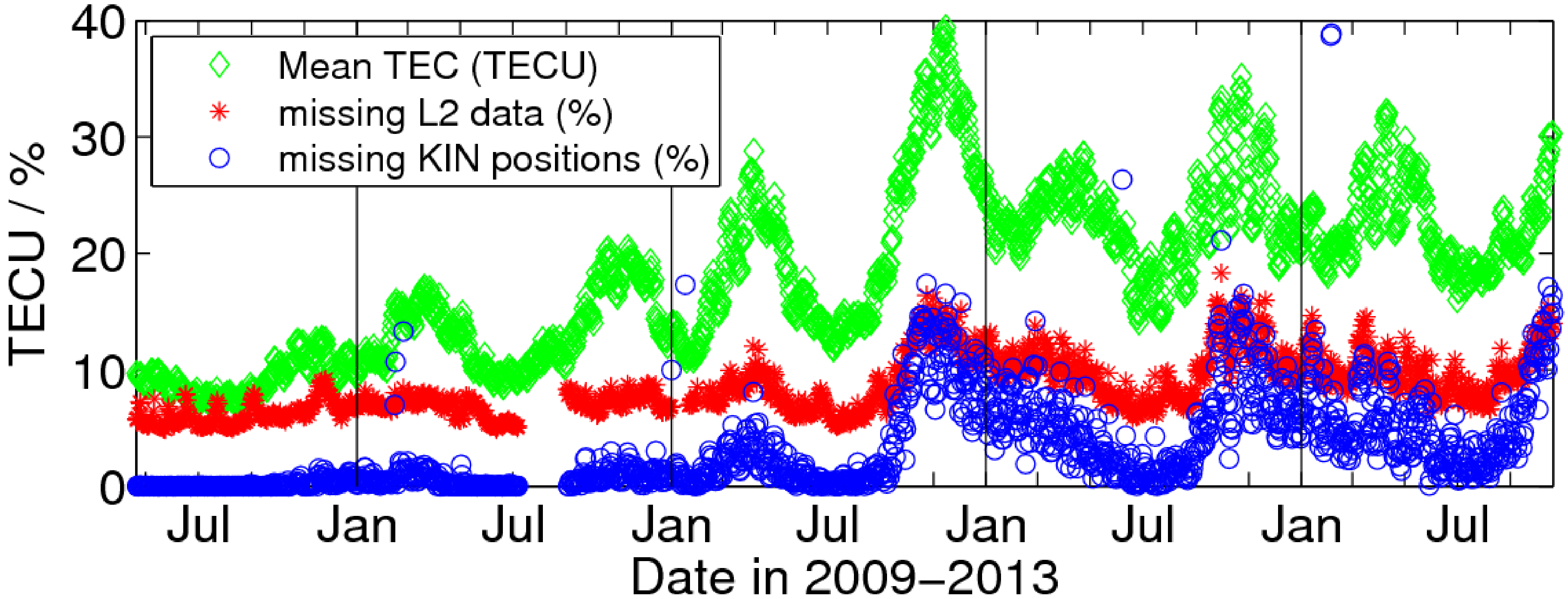
Orbit Differences KIN-RD (COSMIC-2)

Differences at epochs of kin. positions, FM-1, POD-1



Comparison of ambiguity-float solutions and ambiguity-fixed solutions.

Orbit Differences KIN-RD (GOCE: entire mission)



The result illustrates the **consistency** between both orbit-types. The level of the differences is usually given by the quality of the kinematic positions.

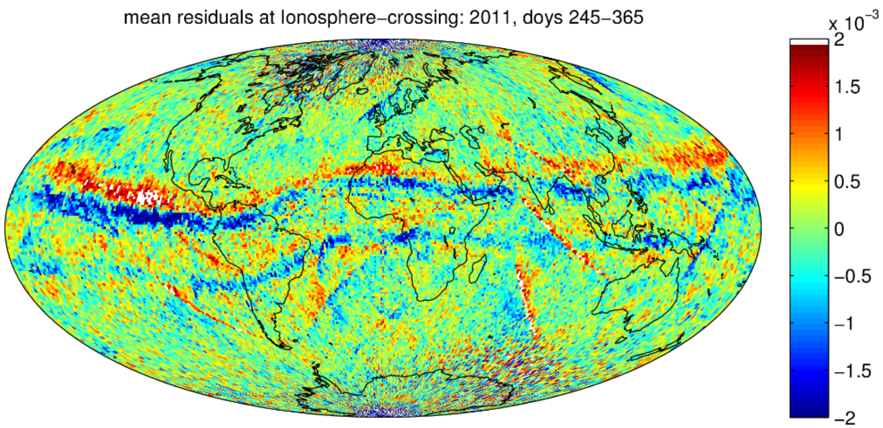
The differences are highly correlated with the **ionosphere activity** and with **data losses on L2**.

(Bock et al., 2014)

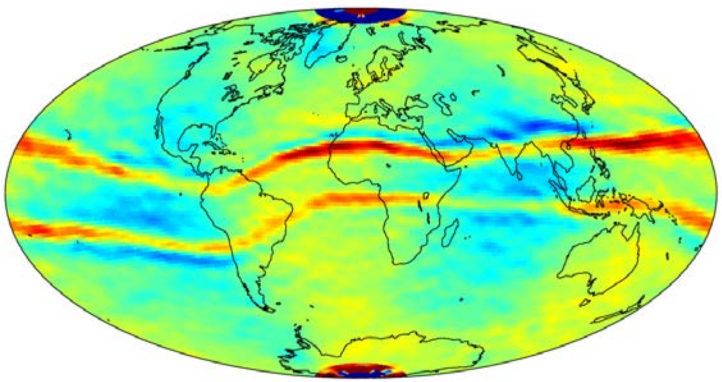
Consequences of Ionospheric Effects in Orbits

For GOCE systematic effects around the geomagnetic equator were observed in the ionosphere-free GPS phase residuals **=> affects kinematic positions**

Degradation of kinematic positions around the geomagnetic equator propagates into gravity field solutions.



Phase observation residuals (- 2 mm … +2 mm) mapped to the ionosphere piercing point

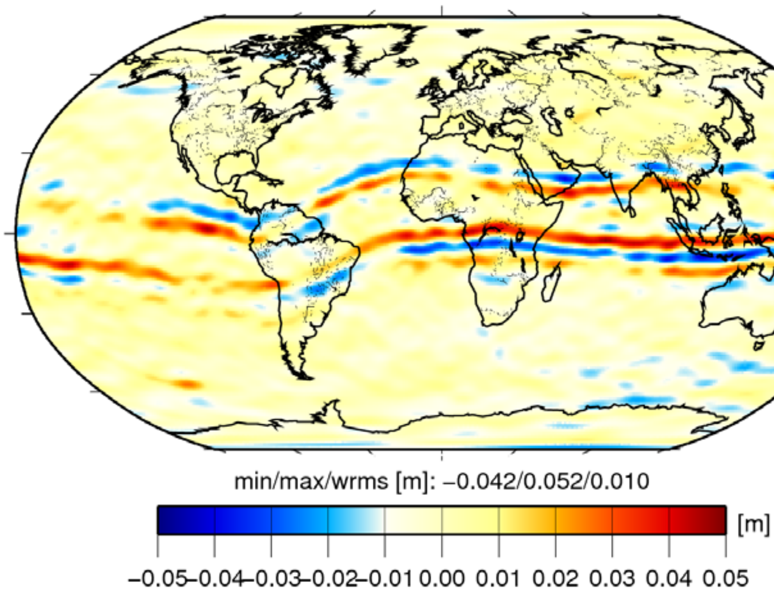


Geoid height differences (−5 cm … 5 cm);
R4 period

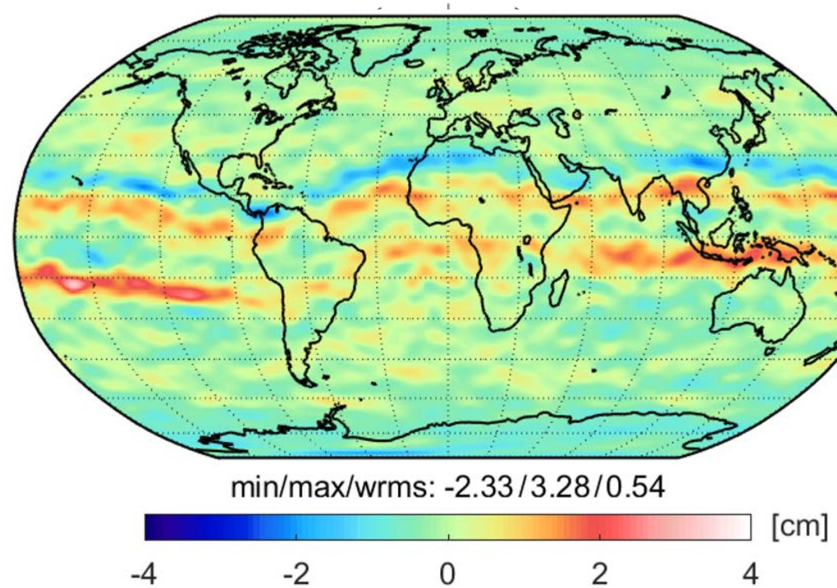
(Jäggi et al., 2015)

Systematic Errors in GPS Data (1)

Original GPS Data
(Swarm)



Original GPS Data
(GRACE)

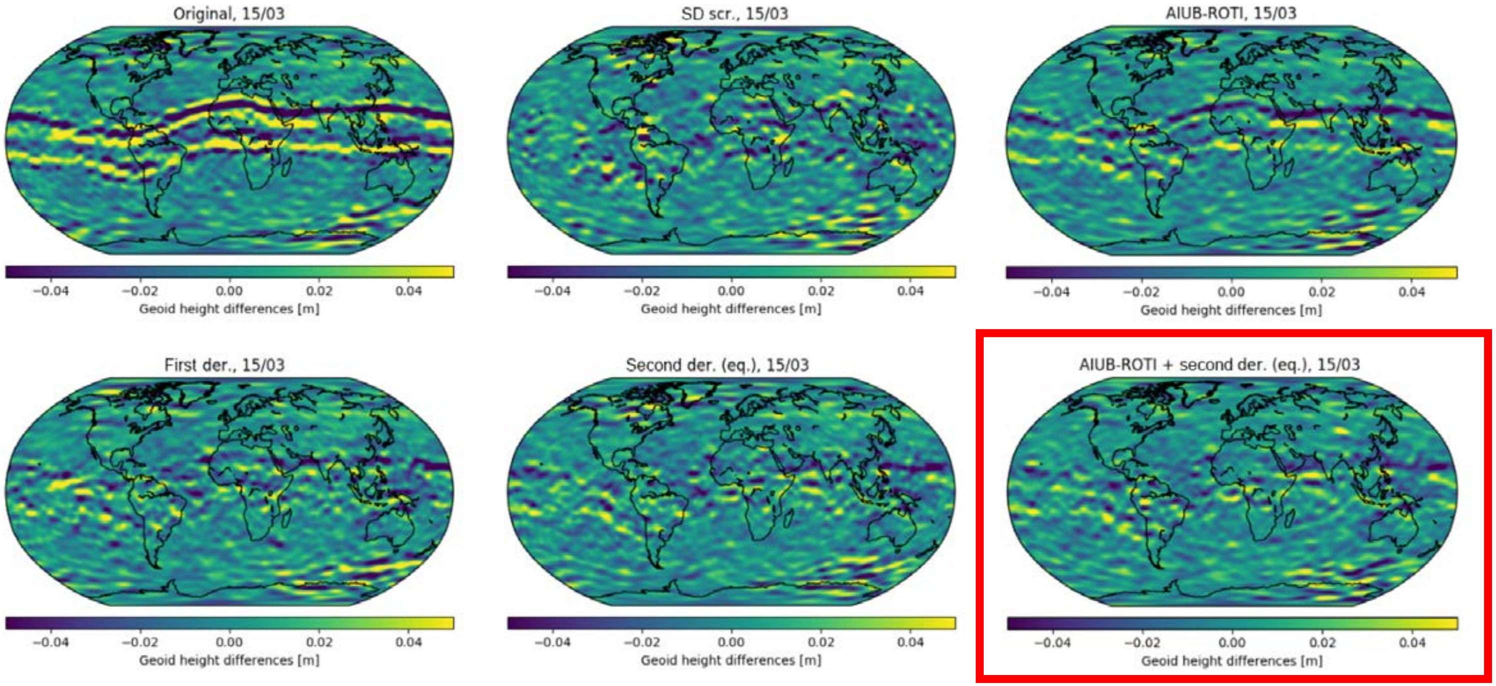


(Differences wrt GOCO05S, 400 km Gauss smoothing adopted)

Systematic signatures along the geomagnetic equator are **"not"** visible when using original L1B RINEX GPS data files from the GRACE mission.

(Jäggi et al., 2016)

Systematic Errors in GPS Data (2)

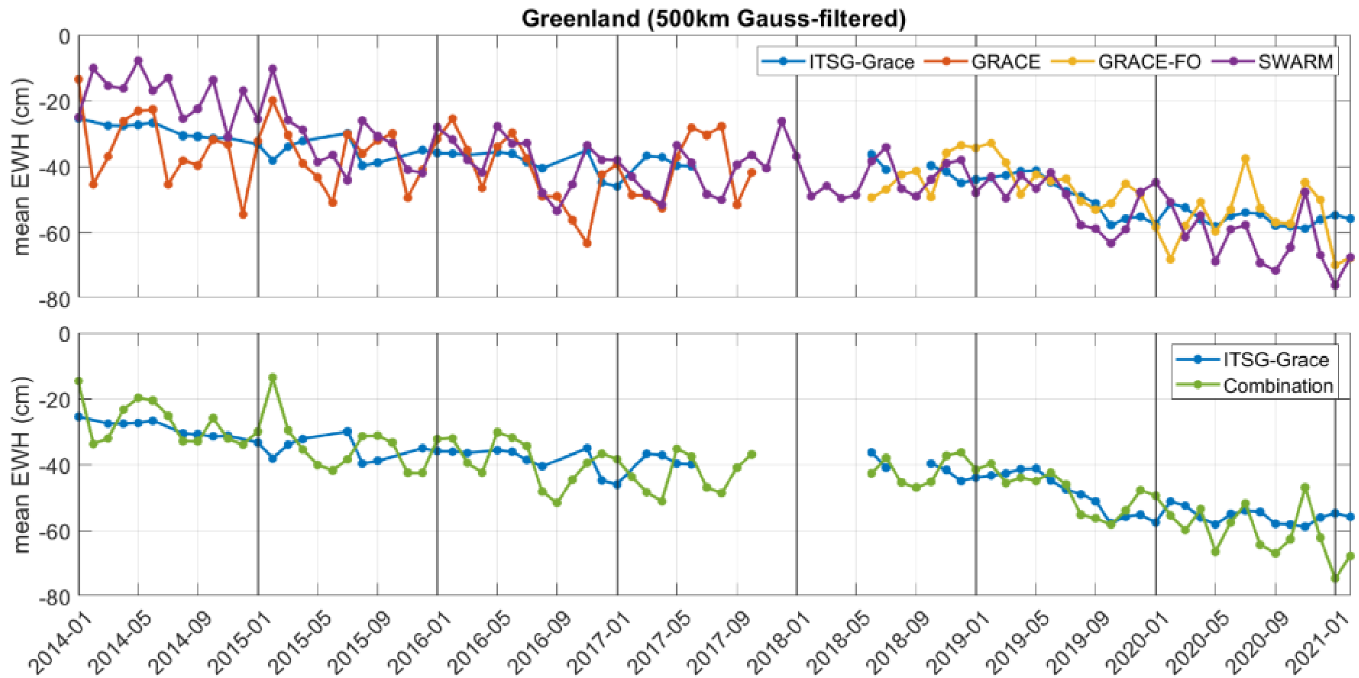


(Differences wrt JPL-GRACE-RL06, 400 km Gauss smoothing adopted)

Systematic signatures along the geomagnetic equator may be efficiently reduced when down-weighting the GPS data using derivatives of the geometry-free linear combination. ROTI-based down-weighting additionally reduces scintillation noise.

(Schreiter et al., 2019)

Recovery of Large-Scale Time-Variable Gravity Signals



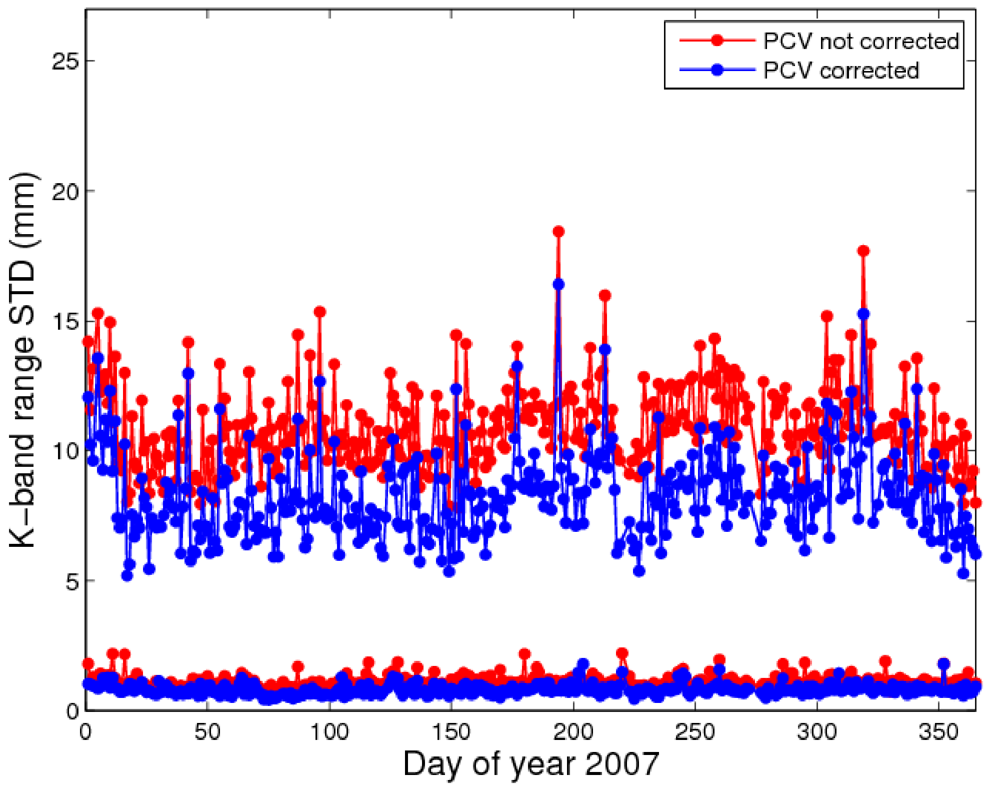
(Time-variable gravity signal up to d/o 40 of the Greenland ice sheet)

Kinematic positions allow to recover the long wavelength part of the Earth's gravity field. Although the scatter is significantly larger than from dedicated GRACE and GRACE-FO data, the trend information may still be recovered remarkably well.

(Grombein et al., 2021)

Orbit Validation

GRACE Orbit Validation with K-Band



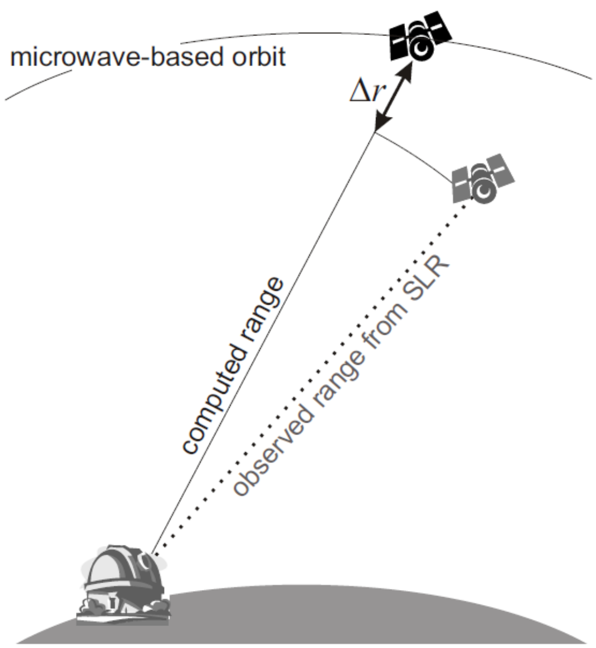
ZD solution
< 1cm

DD solution
(ambiguity-fixed)
< 1mm

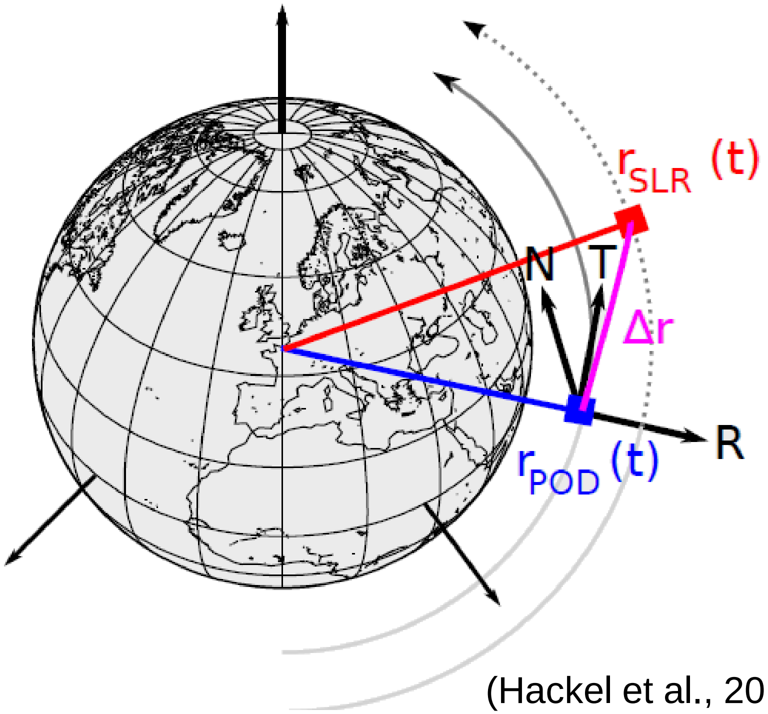
The ultra-precise and continuously available K-Band data allow it to validate the **inter-satellite distances** between the GRACE satellites. Thanks to this validation, e.g., PCV maps were recognized to be crucial for high-quality POD.

(Jäggi et al., 2009)

SLR validation concepts



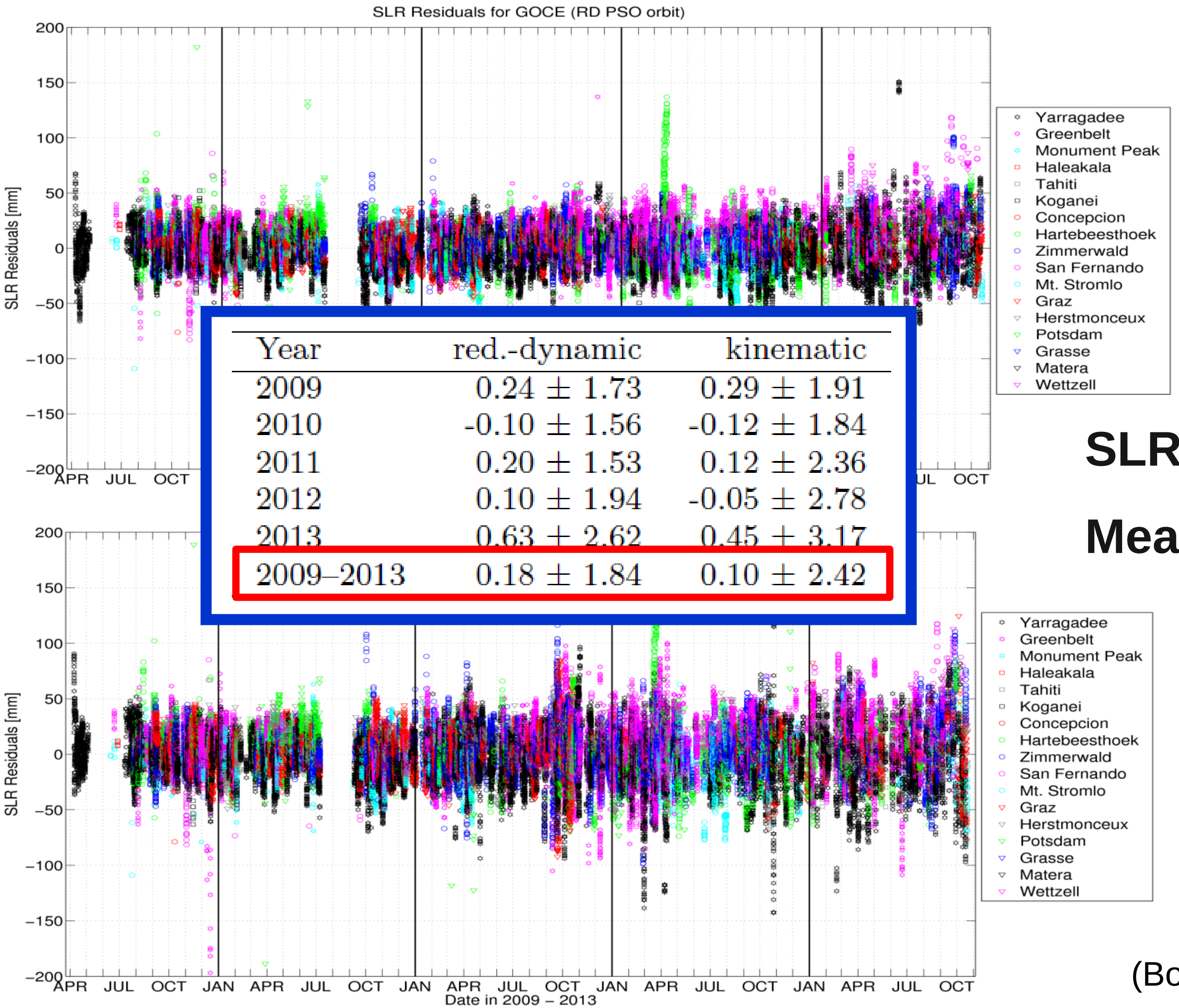
(Flohrer, 2008)



(Hackel et al., 2019)

Especially LEO satellites at **low orbital altitudes** allow not only for a validation of the orbit quality in the radial direction, but also in the other directions. Using long data spans, mean SLR biases may be determined in along-track and cross-track, as well.

Orbit Validation with SLR (GOCE)



Year	red.-dynamic	kinematic
2009	0.24 ± 1.73	0.29 ± 1.91
2010	-0.10 ± 1.56	-0.12 ± 1.84
2011	0.20 ± 1.53	0.12 ± 2.36
2012	0.10 ± 1.94	-0.05 ± 2.78
2013	0.63 ± 2.62	0.45 ± 3.17
2009–2013	0.18 ± 1.84	0.10 ± 2.42

Reduced-dynamic

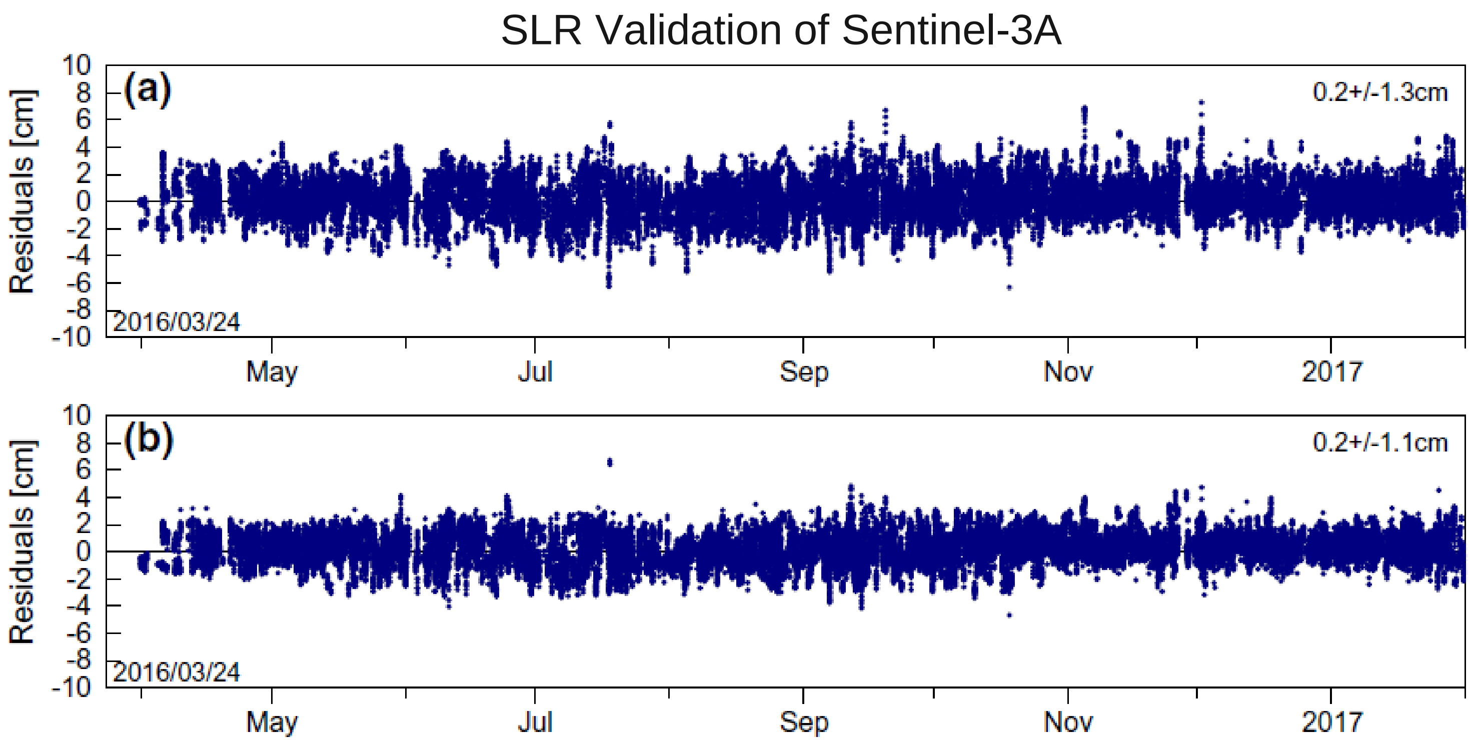
SLR statistics:

Mean ± RMS (cm)

Kinematic

(Bock et al., 2014)

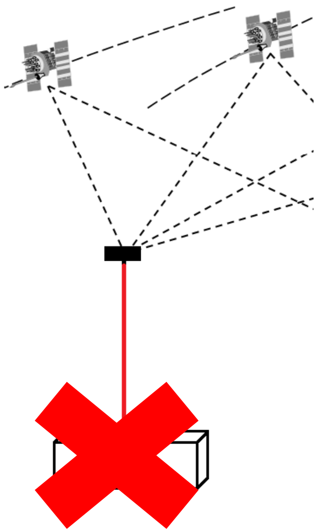
Impact of Undifferenced Ambiguity Resolution (1)

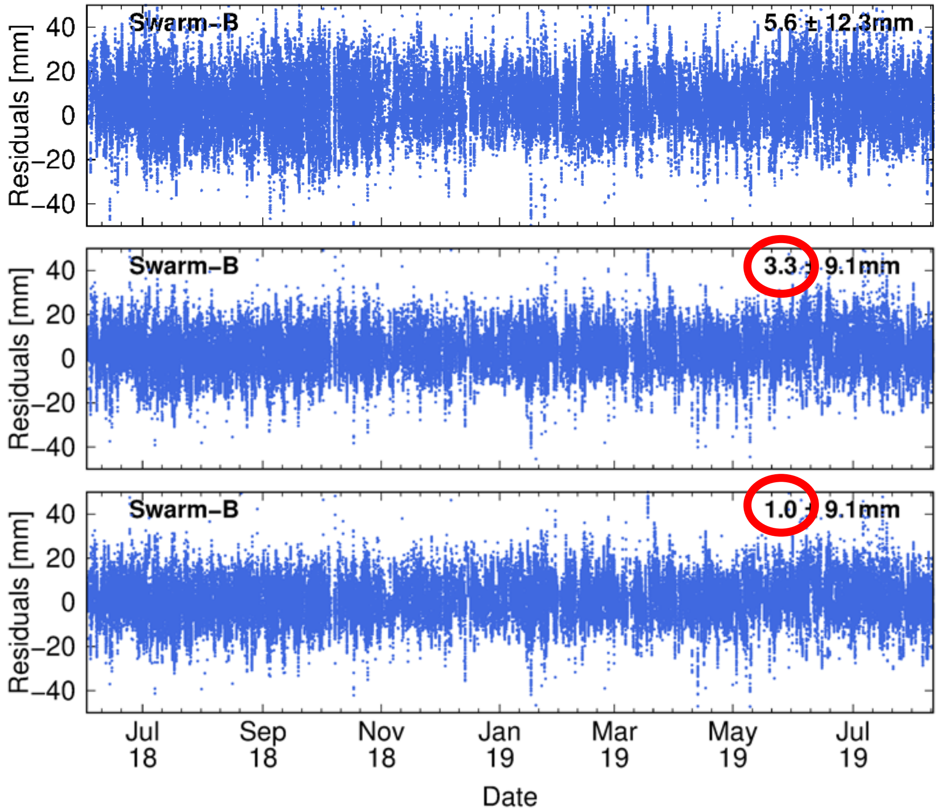


Single-receiver ambiguity fixing may be enabled by using phase bias products and corresponding clock products provided by the IGS analysis centers without the need to form any baselines. It allows to identify lateral offsets in the GPS antenna or center-of-mass location and to significantly stabilize the LEO trajectories.

(Montenbruck et al., 2018)

Impact of Undifferenced Ambiguity Resolution (2)





Ambiguity-float, no non-grav. modeling

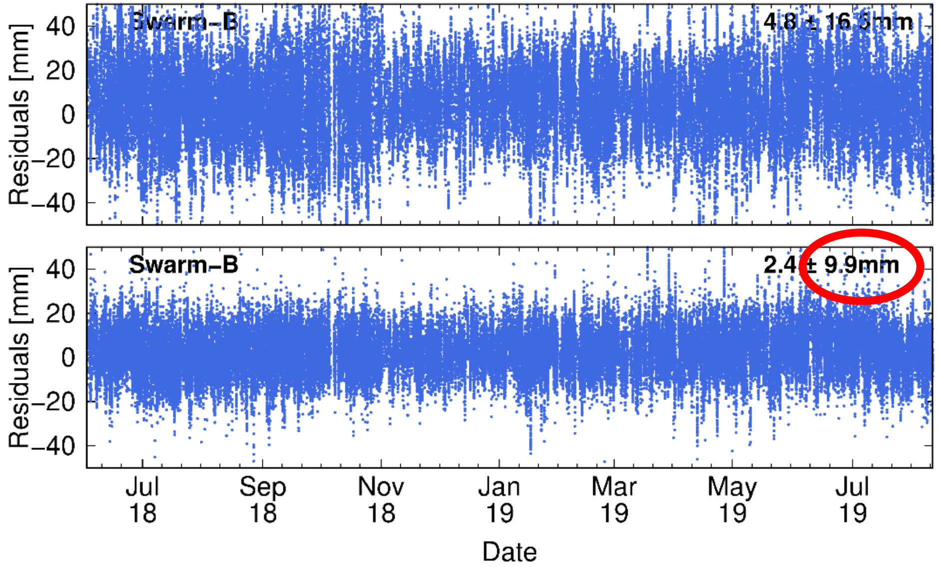
Ambiguity-fixed, no non-grav. modeling

Ambiguity-fixed, with non-grav. modeling

LEO POD significantly profits from single-receiver ambiguity fixing techniques and high-quality signal-specific phase bias products, e.g., by Schaer et al. (2021).

(Arnold et al., 2019; Mao et al., 2021)

Impact of Undifferenced Ambiguity Resolution (3)



Kinematic, ambiguity-float

Kinematic, ambiguity-fixed

The SLR STD of ambiguity-fixed kinematic orbits (9.9mm) is only marginally worse than for the ambiguity-fixed dynamic orbits (9.1mm, see previous slide).

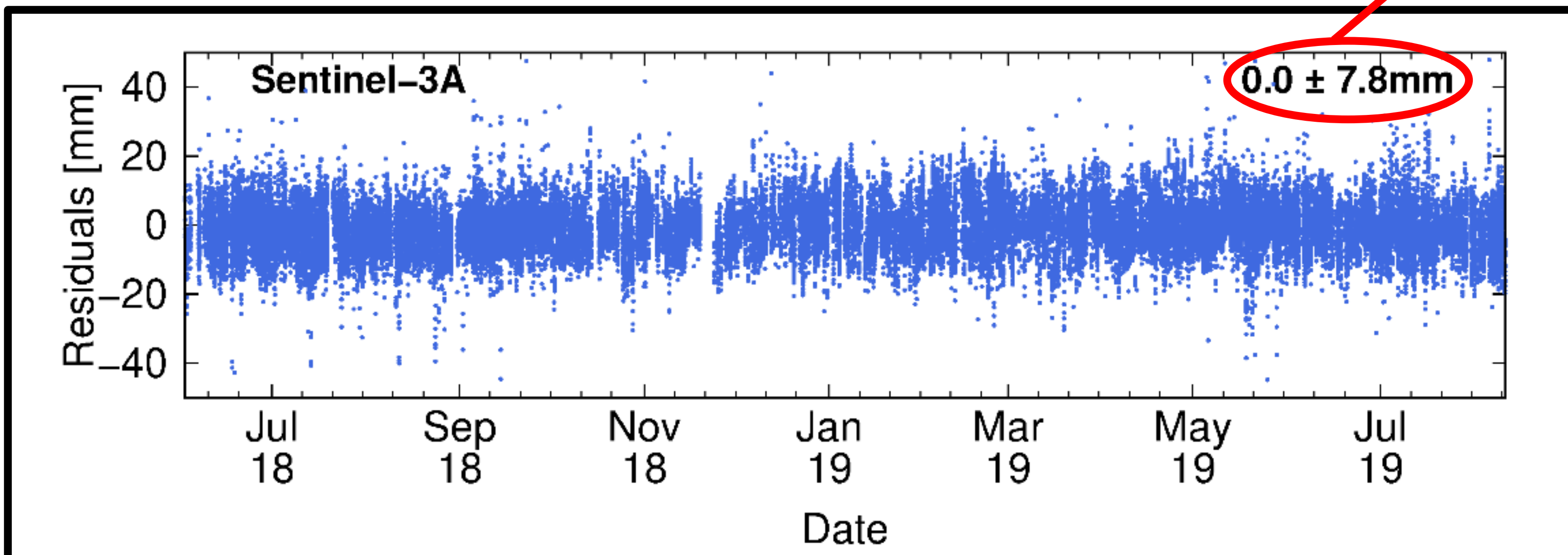
This nicely illustrates the **limitation of SLR** to “distinguish” between the orbits.

Comparisons to ambiguity-fixed kinematic orbits should be regularly performed to detect inconsistencies, e.g., related to wrong GPS antenna phase center offsets.

Orbit Validation – or SLR Network Validation?

Corrections from 1-year of dynamic, ambiguity-fixed Swarm-A/B/C, Sentintel-3A/B and GRACE-FO-C/D orbits.

compared to 1.5 ± 10.5 mm



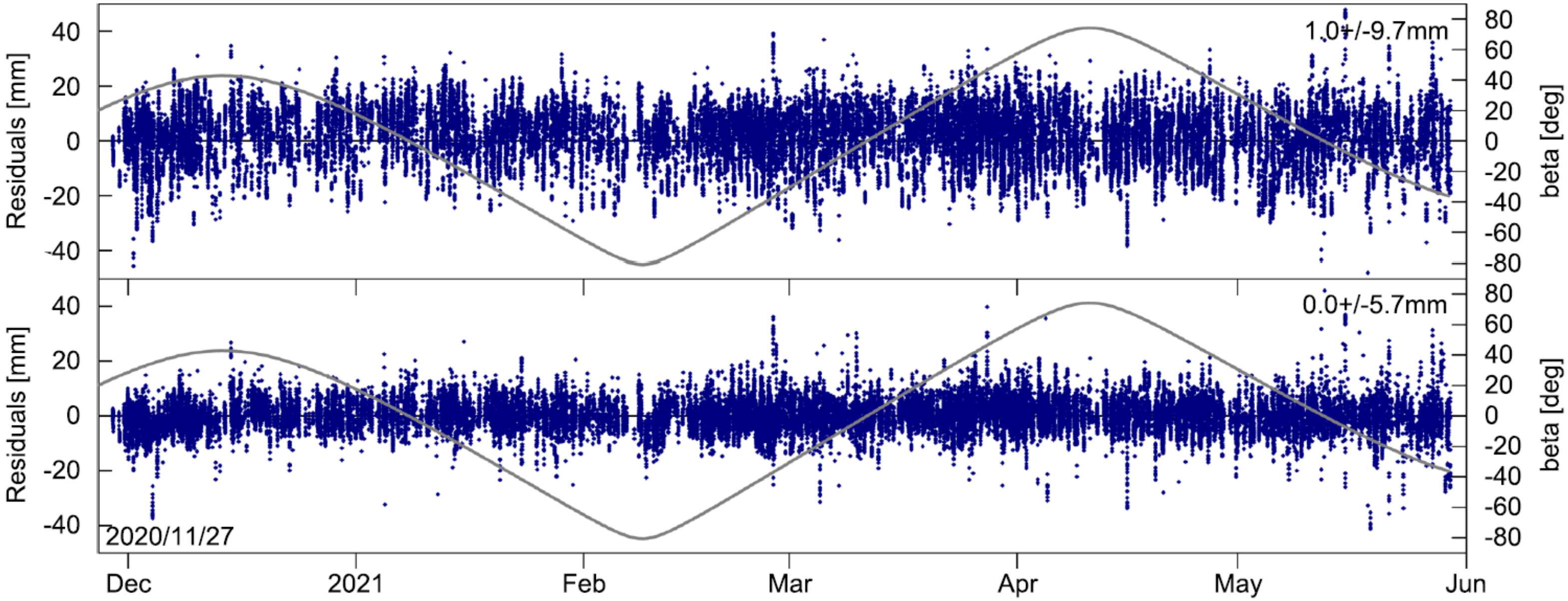
Graz	78393402	2.7 ± 0.2	3.2 ± 0.2	8.7 ± 0.7	11.9 ± 0.4
Herstmonceux	78403501	3.1 ± 0.3	1.5 ± 0.3	-4.1 ± 1.0	-2.5 ± 0.6
Potsdam	78418701	0.9 ± 0.3	3.7 ± 0.3	17.1 ± 0.9	-0.6 ± 0.6
Matera	79417701	1.7 ± 0.4	4.8 ± 0.4	4.2 ± 2.0	-5.3 ± 1.0

Some larger corrections ask for further investigations, e.g. comparisons to LAGEOS-based coordinate solutions. Investigations are on-going …

(Arnold et al., 2019)

Outlook: Multi-GNSS LEO POD

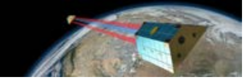
SLR Validation of Sentinel-6A



Station offsets adjusted

While **Galileo** measurements **exhibit 30–50% smaller RMS errors** than those of GPS, the POD benefits most from the availability of an increased number of satellites. For Sentinel-6A a 1-cm consistency of ambiguity-fixed GPS-only and Galileo-only solutions with the dual-constellation orbits can be demonstrated.

(Montenbruck et al., 2021)



Literature (1)

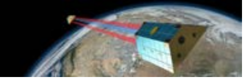
Arnold, D., A. Jäggi, S. Schaer, C. Kobel, U. Meyer, L. Geisser, O. Montenbruck (2019): Performance of dynamic and ambiguity-fixed LEO orbits in SLR validation and network calibration. OSTST 2019 Meeting, October 21-25, 2019.

Beutler, G. (2005) Methods of Celestial Mechanics. Vol 1: Physical, Mathematical, and Numerical Principles. Springer, ISBN 3-540-40749-9

Blewitt, G. (1997): Basics of the GPS Technique: Observation Equations, in *Geodetic Applications of GPS*, Swedish Land Survey, pp. 10-54, available at http://www.sbg.ac.at/mat/staff/revers/lectures/2006_2007/GPS/GPSBasics.pdf

Bock, H., R. Dach, A. Jäggi, G. Beutler (2009): High-rate GPS clock corrections from CODE: Support of 1 Hz applications. *Journal of Geodesy*, 83(11), 1083-1094, doi: 10.1007/s00190-009-0326-1

Bock, H., A. Jäggi, G. Beutler, U. Meyer (2014): GOCE: Precise orbit determination for the entire mission. *Journal of Geodesy*, 88(11), 1047-1060, doi: 10.1007/s00190-014-0742-8



Literature (2)

Dach, R., E. Brockmann, S. Schaer, G. Beutler, M. Meindl, L. Prange, H. Bock, A. Jäggi, L. Ostini (2009): GNSS processing at CODE: status report, *Journal of Geodesy*, 83(3-4), 353-366, doi: 10.1007/s00190-008-0281-2

Flohrer, C. (2008): Mutual Validation of Satellite-Geodetic Techniques and its Impact on GNSS Orbit Modeling. *Geodätisch-geophysikalische Arbeiten in der Schweiz*, 75, Schweizerische Geodätische Kommission, available at http://www.sgc.ethz.ch/sgc-volumes/sgk-75.pdf

Grombein, T., M. Lasser, D. Arnold, U. Meyer, A. Jäggi (2021): Determination and combination of gravity field time series derived from kinematic positions of GRACE/GRACE-FO and Swarm. Submitted to *IAG Symposia Series*

Hackel, S. (2019): Refinement of Reduced-Dynamic Orbit Determination for Low Earth Satellites. *Dissertation der Technischen Universität München*, available at http://mediatum.ub.tum.de/?id=1471591

Jäggi, A., U. Hugentobler, G. Beutler (2006): Pseudo-stochastic orbit modeling techniques for low-Earth satellites. *Journal of Geodesy*, 80(1), 47-60, doi: 10.1007/s00190-006-0029-9

Literature (3)

Jäggi, A. (2007): Pseudo-Stochastic Orbit Modeling of Low Earth Satellites Using the Global Positioning System. *Geodätisch-geophysikalische Arbeiten in der Schweiz*, 73, Schweizerische Geodätische Kommission, available at http://www.sgc.ethz.ch/sgc-volumes/sgk-73.pdf

Jäggi, A., R. Dach, O. Montenbruck, U. Hugentobler, H. Bock, G. Beutler (2009): Phase center modeling for LEO GPS receiver antennas and its impact on precise orbit determination. *Journal of Geodesy*, 83(12), 1145-1162, doi: 10.1007/s00190-009-0333-2

Jäggi, A., L. Prange, U. Hugentobler (2011): Impact of covariance information of kinematic positions on orbit reconstruction and gravity field recovery. *Advances in Space Research*, 47(9), 1472-1479, doi: 10.1016/j.asr.2010.12.009

Jäggi, A., H. Bock, U. Meyer, G. Beutler, J. van den IJssel (2015): GOCE: assessment of GPS-only gravity field determination. *Journal of Geodesy*, 89(1), 33-48. doi: 10.1007/s00190-014-0759-z

Jäggi, A., C. Dahle, D. Arnold, H. Bock, U. Meyer, G. Beutler, J. van den IJssel (2016): Swarm kinematic orbits and gravity fields from 18 months of GPS data. *Advances in Space Research*, 57(1), 218-233. doi: 10.1016/j.asr.2015.10.035

Literature (4)

Jäggi, A., C. Dahle, D. Arnold, H. Bock, U. Meyer, G. Beutler, J. van den IJssel (2016): Swarm kinematic orbits and gravity fields from 18 months of GPS data. *Advances in Space Research*, 57(1), 218-233. doi: 10.1016/j.asr.2015.10.035

Lasser, M., U. Meyer, A. Jäggi, T. Mayer-Gürr, A. Kvas, K.H. Neumayer, C. Dahle, F. Flechtner, J.-M. Lemoine, I. Koch, M. Weigelt, J. Flury (2020): Benchmark data for verifying background model implementations in orbit and gravity field determination software. *Advances in Geosciences*, 55, 1-11. doi: 10.5194/adgeo-55-1-2020

Mao, X., D. Arnold, V. Girardin, A. Villiger, A. Jäggi (2021): Dynamic GPS-based LEO orbit determination with 1 cm precision using the Bernese GNSS Software. *Advances in Space Research*, 67(2), 788-805. doi: 10.1016/j.asr.2020.10.012

Mayer-Gürr, T., A. Kvas (2019): COST-G software comparison, Graz University of Technology, benchmark data set available at: ftp://ftp.tugraz.at/outgoing/ITSG/COST-G/softwareComparison/

Montenbruck, O., S. Hackel, A. Jäggi (2018): Precise orbit determination of the Sentinel-3A altimetry satellite using ambiguity-fixed GPS carrier phase observations. *Journal of Geodesy*, 92(7), 711-726. doi: 10.1007/s00190-017-1090-2

Literature (5)

Montenbruck, O., S. Hackel, M. Wermuth, F. Zangerl (2021): Sentinel-6A precise orbit determination using a combined GPS/Galileo receiver. Journal of Geodesy, Geod 95, 109. doi: 10.1007/s00190-021-01563-z

Schaer, S., A. Villiger, D. Arnold, R. Dach, L. Prange, A. Jäggi (2021): The CODE ambiguity-fixed clock and phase bias analysis products: generation, properties, and performance. *Journal of Geodesy*, 95, 81. doi: 10.1007/s00190-021-01521-9

Schreiter, L., D. Arnold, V. Sterken, A. Jäggi (2019): Mitigation of ionospheric signatures in Swarm GPS gravity field estimation using weighting strategies. *Annales Geophysicae*, 37, 111-127. doi: 10.5194/angeo-37-111-2019

Svehla, D., M. Rothacher (2004): Kinematic Precise Orbit Determination for Gravity Field Determination, in *A Window on the Future of Geodesy*, edited by F. Sanso, pp. 181-188, Springer, doi: 10.1007/b139065

Teunissen, P.J.G., O. Montenbruck (2017): Springer Handbook of Global Navigation Satellite Systems, Springer, ISBN: 978-3-030-73172-4

Pocket Guide of Least-Squares Adjustment (1)

The system of **Observation Equations** is given by:

$$\boldsymbol{L}' + \boldsymbol{\epsilon} = \boldsymbol{F}(\boldsymbol{X})$$

or, if $\boldsymbol{F}$ is a non-linear function of the parameters, in its **linearized** form:

$$\boldsymbol{L}' + \boldsymbol{\epsilon} = \boldsymbol{F}(\boldsymbol{X}_0) + \boldsymbol{A}\,\boldsymbol{x}$$

$\boldsymbol{L}'$	Tracking observations	$\boldsymbol{X}_0$	A priori parameter values
$\boldsymbol{\epsilon}$	Observation corrections	$\boldsymbol{x}$	Parameter corrections
$\boldsymbol{F}$	Functional model	$\boldsymbol{X}$	Improved parameter values, i.e., $\boldsymbol{X} = \boldsymbol{X}_0 + \boldsymbol{x}$

$$\boldsymbol{A} \doteq \left.\frac{\partial \boldsymbol{F}(\boldsymbol{X})}{\partial \boldsymbol{X}}\right|_{\boldsymbol{X}=\boldsymbol{X}_0}$$

First design matrix

Pocket Guide of Least-Squares Adjustment (2)

The system of **Normal Equations** is obtained by minimizing $\boldsymbol{\epsilon}^T \, P \, \boldsymbol{\epsilon}$:

$$\left(\boldsymbol{A}^T \, \boldsymbol{P} \, \boldsymbol{A}\right) \, \boldsymbol{x} - \boldsymbol{A}^T \, \boldsymbol{P} \, \boldsymbol{l} = \boldsymbol{N} \, \boldsymbol{x} - \boldsymbol{b} = \boldsymbol{0}$$

$\boldsymbol{N} \doteq \boldsymbol{A}^T \, \boldsymbol{P} \, \boldsymbol{A}$	Normal equation matrix
$\boldsymbol{b} \doteq \boldsymbol{A}^T \, \boldsymbol{P} \, \boldsymbol{l}$	Right-hand side with "O-C" term $\boldsymbol{l} \doteq \boldsymbol{L}' - \boldsymbol{F}(\boldsymbol{X}_0)$
$\boldsymbol{P} = \sigma_0^2 \, \boldsymbol{C}_{\boldsymbol{ll}}^{-1}$	Weight matrix, from covariance matrix $\boldsymbol{C}_{\boldsymbol{ll}}$ of observations

For a **regular** normal equation matrix the parameter corrections follow as:

$$\boldsymbol{x} = \left(\boldsymbol{A}^T \, \boldsymbol{P} \, \boldsymbol{A}\right)^{-1} \, \boldsymbol{A}^T \, \boldsymbol{P} \, \boldsymbol{l} = \boldsymbol{N}^{-1} \, \boldsymbol{b}$$

Pocket Guide of Least-Squares Adjustment (3)

The **a posteriori standard deviation of unit weight** is computed as:

$$m_0 = \sqrt{\frac{\boldsymbol{\epsilon}^T \, P \, \boldsymbol{\epsilon}}{f}}$$

f Degree of freedom (number of observations minus number of parameters)

The **covariance matrix** of the adjusted parameters is given by

$$\boldsymbol{C_{xx}} = m_0^2 \, \boldsymbol{Q_{xx}} = m_0^2 \, \boldsymbol{N}^{-1}$$

and their a posteriori standard deviations follow from the diagonal elements:

$$m_x = \sqrt{C_{xx}} = m_0 \, \sqrt{Q_{xx}}$$

Pocket Guide of Least-Squares Adjustment (4)

Parameter pre-elimination is useful to handle a large number of parameters efficiently. Let us sub-divide the system of normal equations into two parts:

$$\begin{pmatrix} \boldsymbol{N}_{11} & \boldsymbol{N}_{12} \\ \boldsymbol{N}_{21} & \boldsymbol{N}_{22} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$$

We we may reduce the normal equation system by pre-eliminating epoch-specific parameters $\boldsymbol{x}_2$, which yields the modified system of normal equations as

$$\boldsymbol{N}_{11}^* \, \boldsymbol{x}_1 = \boldsymbol{b}_1^*$$

where

$\boldsymbol{N}_{11}^* = \boldsymbol{N}_{11} - \boldsymbol{N}_{12} \, \boldsymbol{N}_{22}^{-1} \, \boldsymbol{N}_{21}$ is the normal equation matrix of $\boldsymbol{x}_1$

$\boldsymbol{b}_1^* = \boldsymbol{b}_1 - \boldsymbol{N}_{12} \, \boldsymbol{N}_{22}^{-1} \, \boldsymbol{b}_2$ is the corresponding right-hand side of the normal equation system