Localisation précise par moyens spatiaux

Parameter estimation in Celestial Mechanics

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Most of the material used in this part of the course stems from Beutler (2005), Chapter 8. Section 6 in essence relies on Beutler (2005), Vol. I, Chapter 5.

Orbit determination as parameter estimation

Calculation of ephemerides und *Orbit determination* are the most important tasks of applied astrodynamics (celestial mechanics).

Orbit determination may be viewed as the inverse task of ephemeris calculation.

Whereas the production of ephemerides may be viewed as “pure routine work”, orbit determination is much more more difficult.

In pure orbit determination we have the task

- to find a particular solution of the EQs of motion for a particular CB
- from observations as well as
- To reconstruct the trajectory/trajectories of the observers.

In satellite geodesy one may often end up with with a problem, where thousands of parameters have to be solved.

Today, there are many observation types (astrometric positions, distances, Doppler observations, etc.).

The classical task

From a series of astrometric places

- t_i, α_i', δ_i', $i=1,2,\ldots,n > 2$

(observation times, right ascensions, declinations) the *osculating orbit elements, e.g.,*

- $a, e, i, \Omega, \omega, T_0$

of a CB have to be determined.

The EQs of motion of the CB and the trajectories of the observers are assumed as known. Provided the time interval $[t_1,t_n]$ is short, one may use the EQs of the two-body problem::

$$\ddot{\boldsymbol{r}} = -\mu \cdot \frac{\boldsymbol{r}}{r^3}$$

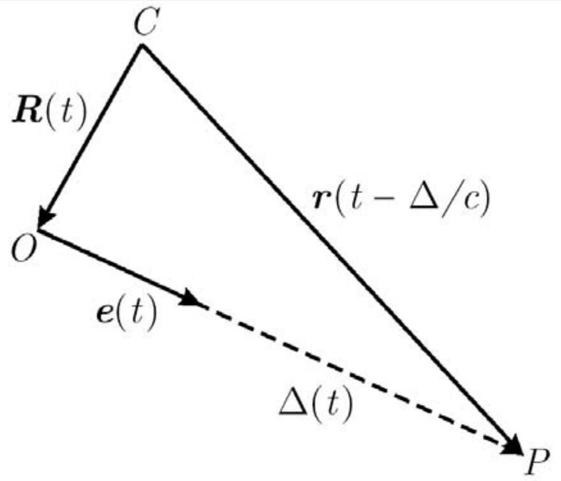
The constant μ depends on the CB. If one has to deal with a (minor) planet or a comet, one has:

$\mu = k^2 = 0.01720209895^2$,

For artificial Earth satellites: $\mu = GM = 398.\ 6004415 \cdot 10^{12}$

The constant has the dimension [length^3/time^2].

The classical task



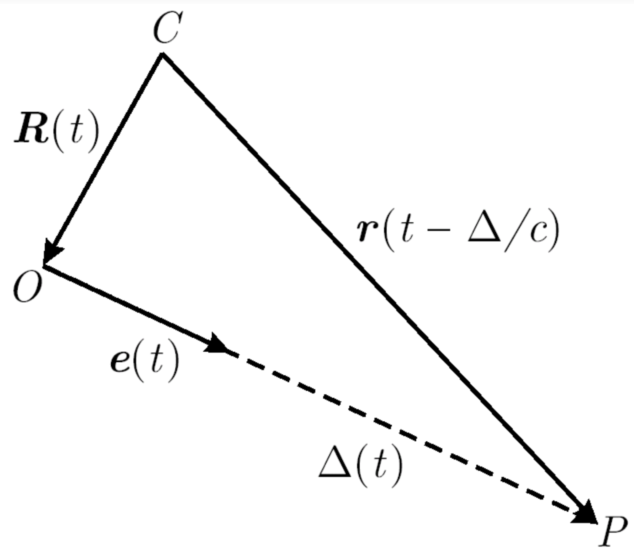
The unit vector ***e****(t)* is defined by the R.A. α and declination δ of the CB. ***e****(t)* defines the astrometric place of the CB at time *t*.

Precisely speaking ***e****(t)* defines the direction from the observer at observation time *t* to the CB at time $t-\Delta/c$, where Δ is the distance from the observer at *t* to the CB at time $t-\Delta/c$.

$c = 299792.458$ km/s is the speed of light.

R*(t)* is the heliocentric position vector (assumed as known).

The classical task



The astrometric place of the CB and the heliocentric positions of CB and observer are related by:

$$\boldsymbol{\Delta}(t) = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = \Delta \begin{pmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{pmatrix} \stackrel{\mathrm{def}}{=} \boldsymbol{r}\left(t - \frac{\Delta}{c}\right) - \boldsymbol{R}(t)$$

The classical task

We make the distinction of

- First orbit determination
- Orbit improvement.

Depending on whether

- approximate orbit elements are not available or available and used.

Today orbit improvement is *routine*, whereas first orbit determination still may contain „artistic“ elements.

We will first deal with routine (orbit determination), then with the fine art of first orbit determination.

We will first treat the orbit as a solution of the two-body problem. Only in the section “variational equations” we will deal with the more general case.

This approximation is usually adequate if the time span covered by observations is a small fraction of the revolution period (this is true in the planetary system and in satellite geodesy,).

Orbit improvement

Orbit improvement:

We assume that a set of approximate orbital elements is known:

$$a^K, e^K, i^K, \Omega^K, \omega^K, \text{ and } T_0^K$$

We develop the *observed functions* into a Taylor series using the above values as origin of the development and truncate the series after the terms of first order:

$$\alpha(t; a, e, i, \Omega, \omega, T_0) = \alpha^K(t) + \sum_{j=1}^{6} \left(\frac{\partial \alpha^K}{\partial I_j} \right)(t)\ (I_j - I_j^K) + O(I_k I_l)$$

$$\delta(t; a, e, i, \Omega, \omega, T_0) = \delta^K(t) + \sum_{j=1}^{6} \left(\frac{\partial \delta^K}{\partial I_j} \right)(t)\ (I_j - I_j^K) + O(I_k I_l)\ .$$

where we used for the sake of convenience:

$$\{I_1, I_2, \ldots, I_6\} \stackrel{\text{def}}{=} \{a, e, i, \Omega, \omega, T_0\}$$

Orbit improvement

Neglecting the terms of order 2 and higher we obtain the following linear observation equations in the increments $\Delta I_j = I_j - I_j^K$

$$\sum_{j=1}^{6} \frac{\partial \alpha_i^K}{\partial I_j} \left(I_j - I_j^K\right) - \left(\alpha_i' - \alpha^K(t_i)\right) = v_{\alpha_i}$$

$$\sum_{j=1}^{6} \frac{\partial \delta_i^K}{\partial I_j} \left(I_j - I_j^K\right) - \left(\delta_i' - \delta^K(t_i)\right) = v_{\delta_i}$$

The quanitities *v..*on the RHS are called the residuals of the adjustment. As there usually are more observations than parameters we have to adopt a criterium to obtain a unique solution, e.g., least squares criterion:

$$\sum_{i=1}^{n} \left\{ \left[\cos\delta_i' \, v_{\alpha_i}\right]^2 + v_{\delta_i}^2 \right\} = \min.$$

Orbit improvement

We replace the residuals by the left-hand sides of the observation equations and take the derivatives of the resulting expression w.r.t. the six parameters Δ_j to obtain the linear normal equation system with six equations and six unknowns:

$$\mathbf{N}^K \, \Delta \boldsymbol{I}^K = \boldsymbol{b}^K$$

where:

$$N_{jk}^K = \sum_{i=1}^{n} \left\{ \cos^2 \delta_i' \, \frac{\partial \alpha_i^K}{\partial I_j} \, \frac{\partial \alpha_i^K}{\partial I_k} + \frac{\partial \delta_i^K}{\partial I_j} \, \frac{\partial \delta_i^K}{\partial I_k} \right\}$$

$$b_j^K = \sum_{i=1}^{n} \left\{ \cos^2 \delta_i' \, \frac{\partial \alpha_i^K}{\partial I_j} \left(\alpha_i' - \alpha^K(t_i) \right) + \frac{\partial \delta_i^K}{\partial I_j} \left(\delta_i' - \delta^K(t_i) \right) \right\}$$

The normal equation system (NEQs) is symmetric, positive-definite and may be solved by the standard procedures of linear algebra.

Orbit improvement

The terms α^K and δ^K are obtained by the standard formulas of the TBP.

We still have to say how to calculate the partial derivatives in the observation equations and the NEQs. Let us first apply the chain rule:

$$\frac{\partial \alpha}{\partial I} = \sum_{k=1}^{3} \frac{\partial \alpha}{\partial \Delta_k} \cdot \frac{\partial \Delta_k}{\partial I} = \sum_{k=1}^{3} \frac{\partial \alpha}{\partial \Delta_k} \cdot \frac{\partial r_k}{\partial I}$$

$$\frac{\partial \delta}{\partial I} = \sum_{k=1}^{3} \frac{\partial \delta}{\partial \Delta_k} \cdot \frac{\partial \Delta_k}{\partial I} = \sum_{k=1}^{3} \frac{\partial \delta}{\partial \Delta_k} \cdot \frac{\partial r_k}{\partial I}$$

The gradiants of the observed angles are obtained as:

$$\nabla_{\Delta} \alpha = \frac{1}{\Delta_1^2 + \Delta_2^2} \begin{pmatrix} -\Delta_2 \\ \Delta_1 \\ 0 \end{pmatrix}$$ and $$\nabla_{\Delta} \delta = \frac{1}{\Delta^2 \sqrt{\Delta_1^2 + \Delta_2^2}} \begin{pmatrix} -\Delta_1 \Delta_3 \\ -\Delta_2 \Delta_3 \\ \Delta_1^2 + \Delta_2^2 \end{pmatrix}$$

Orbit improvement

The partial derivatives of the orbital elements are obtained by taking the partial derivatives of the formulas of the TBP w.r.t. the osculating elements:

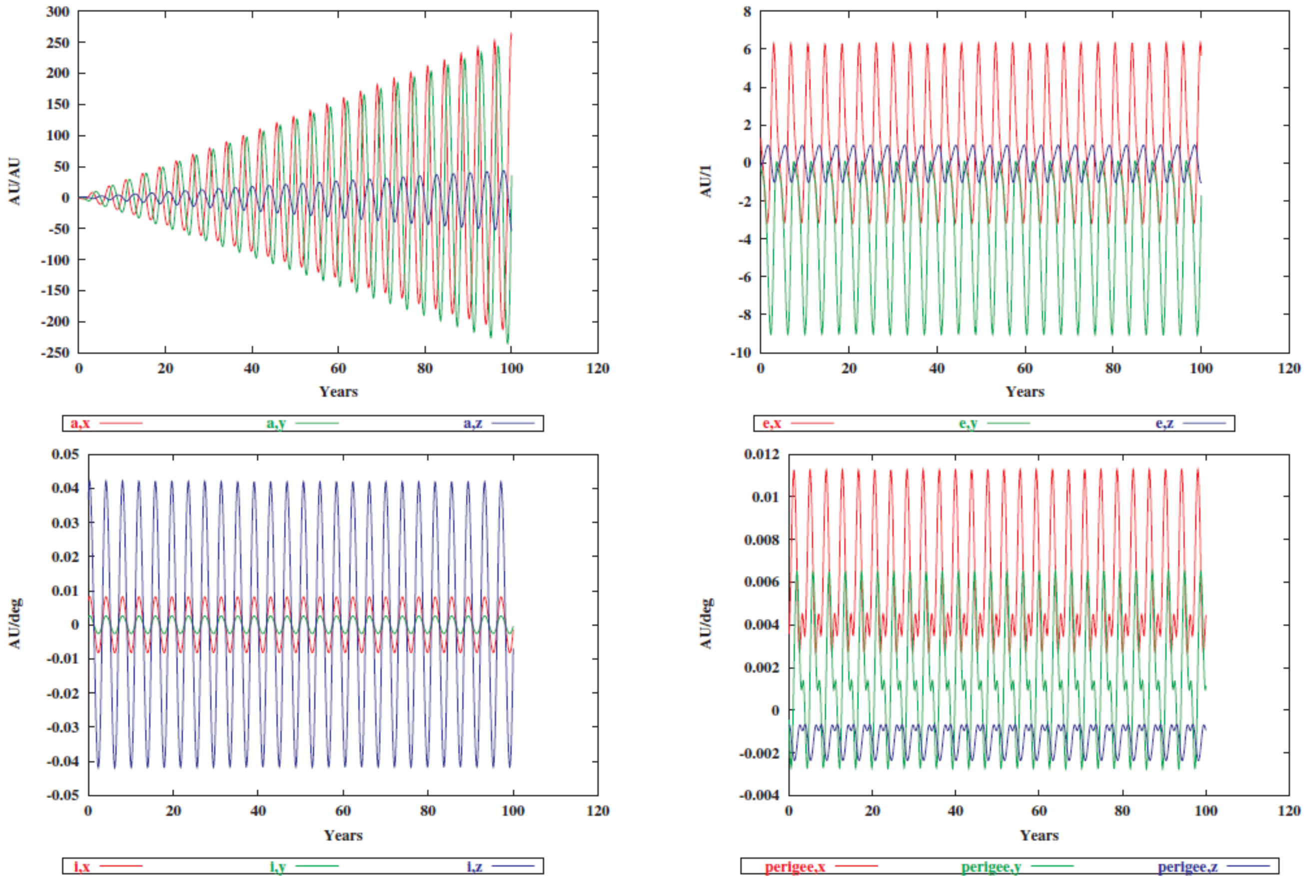
$$\begin{pmatrix} x_e \\ y_e \\ z_e \end{pmatrix} = \mathbf{R}_3(-\Omega)\cdot\mathbf{R}_1(-i)\cdot\mathbf{R}_3(-\omega)\cdot\begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\cdot\sin E \\ 0 \end{pmatrix}$$

For the partial derivative w.r.t. the inclination *i* we obtain, e.g:

$$\frac{\partial}{\partial i}\begin{pmatrix} x_e \\ y_e \\ z_e \end{pmatrix} = \mathbf{R}_3(-\Omega)\cdot\frac{\partial}{\partial i}\left\{\mathbf{R}_1(-i)\right\}\cdot\mathbf{R}_3(-\omega)\cdot\begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\cdot\sin E \\ 0 \end{pmatrix}$$

The other five partial derivatives are formed in an analogous way. For the elements *a* und *e* one has to take into account that they *also* appear in *E* (Kepler's EQ). T_0 only appears in *E*.

Orbit improvement



Partial derivatives of a two-body orbit w.r.t. *a* (left, top), *e* (left, bottom), *I* (right, top) and ω (right, bottom). Minor planet with revolution period of about four years, *e=0.1*, *i=11.58º*. (red: x, green: y, blue: z)

Orbit improvement

Orbit improvement in principle is a non-linear parameter estimation process.

The original, non-linear observation equations read as:

$$t_i\colon \alpha_i - \alpha_i' = v_{\alpha i} \text{ and } \delta_i - \delta_i' = v_{\delta i},\ i = 1,2,..,n$$

The observed functions $\alpha_i(a,e,I,\Omega,\omega,T_0)$ and $\delta_i(a,e,I,\Omega,\omega,T_0)$ have to be linearized, which results in linear observation equations.

The linear(ized) observation equations are solved (if necessary iteratively) using the method of least squares represented by the *least squares criterion* acting on the residuals.

The iterative orbit improvement process may be terminated as soon as the terms of higher the first order in the observation equations are negligible compared to the mean errors of the observations.

Let us add at this point two essential facts related to least squares solutions, namely the rms error a posteriori of errors and the errors of the estimated parameters (other characteristics will be presented in the lecture “advanced parameter estimation”).

Orbit improvement

Leaving out the iteration index "*K*" the mean error a posteriori m_0 of the observations is defined as:

$$m_0 = (((v_{\alpha i} \cos\delta_i')^2 + v_{\delta i}^2))/(2\, n - 6))^{1/2}$$

where *n* is the number of astrometric places (i.e., of pairs $\alpha_i\ \delta_i$).

The NEQs (without superscript "*K*") may by written as:

$$\boldsymbol{N\ \Delta l = b}$$

Its solution may be qiven the form:

$$\boldsymbol{\Delta l = Q\ b}\text{ , where } \boldsymbol{Q = N}^{-1}$$

The matrix ***Q*** is called the co-factor matrix of the adjustment.

The mean error a posteriori of the estimated parameters are:

$$m(\Delta l_j) = m_0\ \mathbf{Q}_{jj}^{1/2}$$

Implying that the diagonal elements of ***Q*** must be positive (matrix ***N*** must be positive-definite).

First Orbit Determination: Circular Orbit

The relationship between orbital elements and observed functgions is non-linear – and remains so in first orbit determination.

The key to the solution of the problems resides in the *reduction of the number of parameters***.**

The principle will be explained using the procedure to determine a circular orbit.

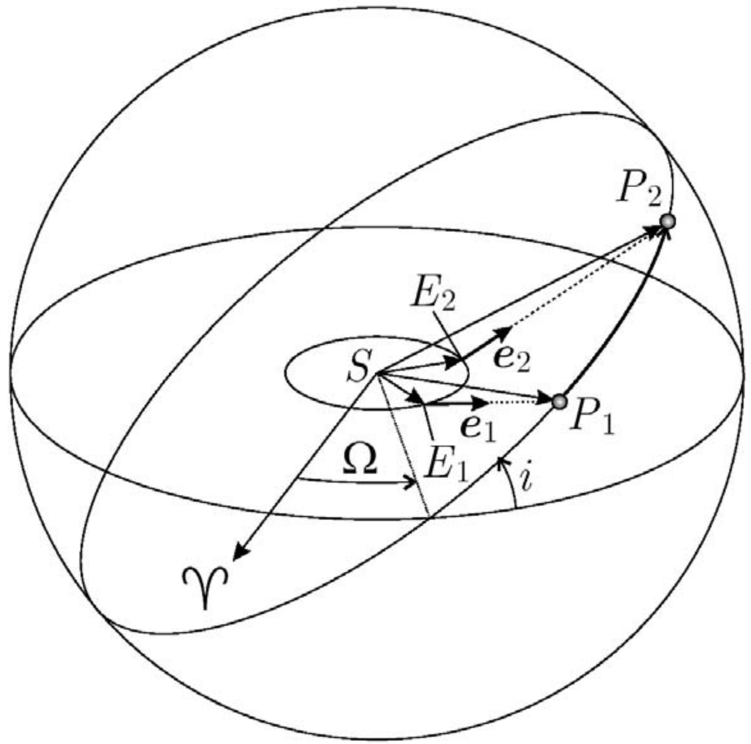
If a (supposedly) new CB is detected one often has only two obser-vation (or more than two in very short time intervals). Determining a circuluar orbit seems appropriate under such circumstances.

The assumption of a circular orbit often makes sense (e.g., for minor planets or for satellites in the geostationary belt).

A circular orbit is defined by only four parameters (instead of six because we may put $e=0$, $\omega=0$).

We may even reduce the problem to find the roots of a scalar function $B(a)$.

First Orbit Determination: Circular Orbit



Assuming a value for *a,* the CB must lie on a heliocentric sphere with radius *a*.

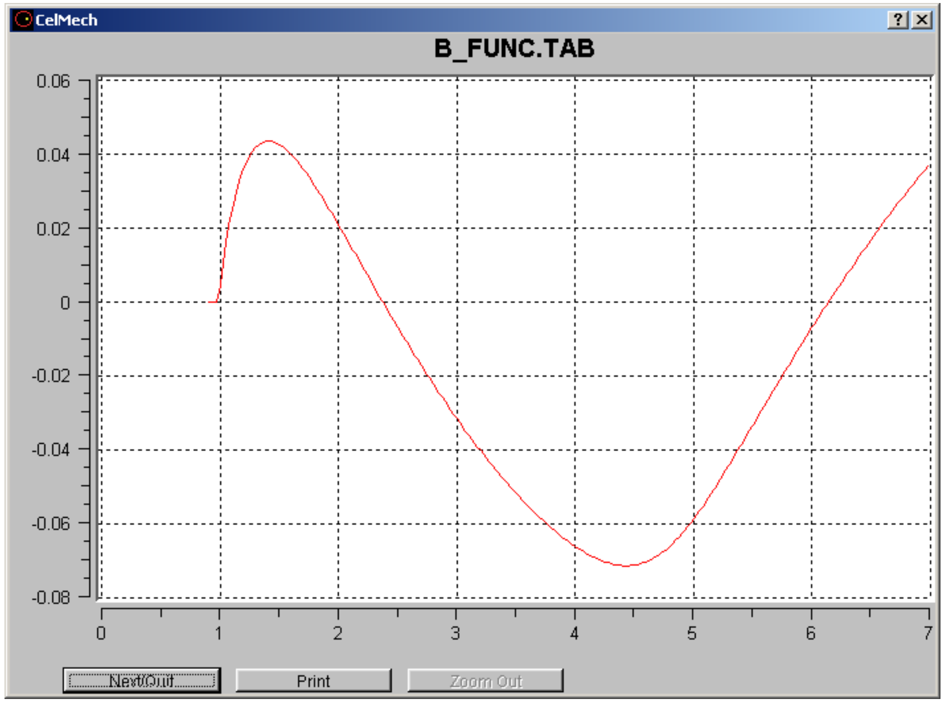
Furthermore the CB must lie at the observation times t_1 and t_2 the on straight lines (rays) defined by the unit vectors $\boldsymbol{e}_1$ und $\boldsymbol{e}_2$.

This implies that the positions $\boldsymbol{P_1}$, $\boldsymbol{P_2}$ of the CB at t_1 , t_2 are known, implying in turn that the angle $\angle(P_1 S\, P_2) =: \boldsymbol{\Delta u_g}$ is known.

For a circular orbit the same angle Δu may be calculated using the law of dynamics $\Delta u_d = n\,(t_2 - t_1)$, where $n = (\,\mu/a^3)^{1/2}$.

Determining a circular orbit is thus equivalent to find the roots of the function $B(a) := \Delta u_g(a) - \Delta u_d(a)$!

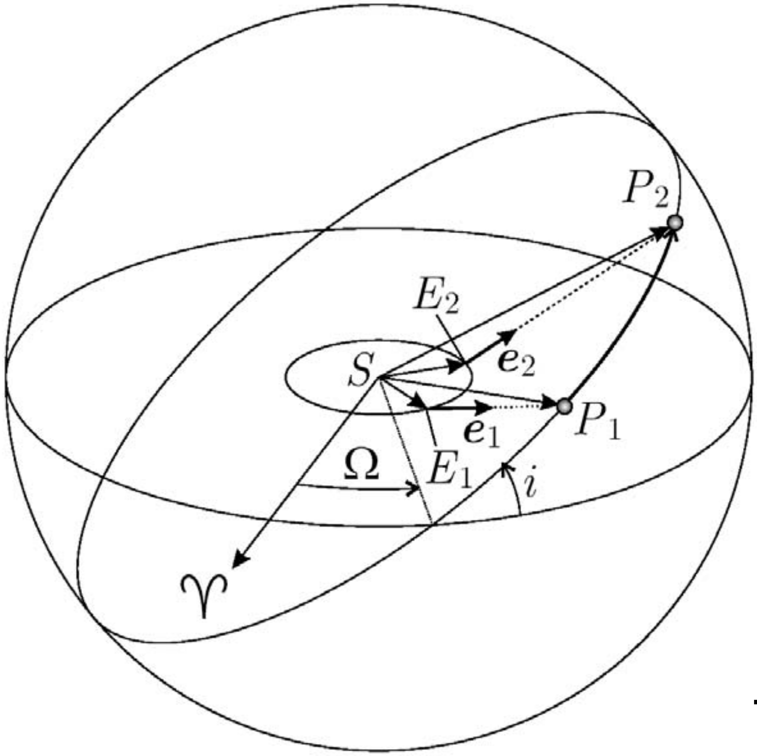
First Orbit Determination: Circular Orbit



$B(a)= \Delta u_g(a) - \Delta u_d(a)$

- Determining a circular orbit of MP *Silentium* with *CelMech,* (using the 3rd and 5th of the observations
- Three solutions are obtained: the first one is the orbit of the observer (?), the third one would result in a retrograde ($i>90^o$). Only the solution at $a=2.4$ remains.
- The resulting elements are
- $a=2.37$ AU, $i=5.79^o$, $\Omega=6.91^o$
- These estimates are close to the "true elements".

Orbit Determination as Boundayry Value Problem



For two observation times we may write

$$\boldsymbol{r}_{b_i} = \boldsymbol{R}_{b_i} + \Delta_{b_i}\, \boldsymbol{e}_{b_i} \ , \quad i = 1, 2$$

Defining the orbit parameters as

$$\{p_1, p_2, \ldots, p_6\} \stackrel{\text{def}}{=} \{\Delta_{b_1}, \Delta_{b_2}, \alpha_{b_1}, \alpha_{b_2}, \delta_{b_1}, \delta_{b_2}\}$$

reduces the number of six orbit parameters to two (the first two).

The two parameters are systematically varied to represent the other observations in the best possible way (LSQ sense).

For each set Δ_{b1} Δ_{b2} RMS a posteriori is calculated. The correct solution minimizes the RMS.

Orbit Determination as Boundayry Value Problem

Adopting values for the topocentric distances at observation times with indices b_1 und b_2, allows it to calculate the the CB's heliocentric positions at these epochs, as wells.

For orbit determination we also need $\boldsymbol{r}(t_i)$ for $i \neq b_1, b_2$. With these heliocentric positions we may calculate the observed functions α_i and δ_i associated with the observations α_i' and δ_i' and the residuals cos δ_i' $(\alpha_i' - \alpha_i)$, $(\delta_i' - \delta_i)$.

By systematically varying the two topocentric distances we obtain the orbit parameters minimizing the sum of the residuals squares.

We thus have to solve the following boundary value problem:

$$\ddot{\boldsymbol{r}} = -\mu \frac{\boldsymbol{r}}{r^3}$$

$$\boldsymbol{r}\left(t_{b_1} - \frac{\Delta_{b_1}}{c}\right) = \boldsymbol{r}_{b_1}$$

$$\boldsymbol{r}\left(t_{b_2} - \frac{\Delta_{b_2}}{c}\right) = \boldsymbol{r}_{b_2}$$

Orbit Determination as Boundayry Value Problem

A general solution of the boundary problem is really difficult. We may, however, make us of the fact that time interval covered by the observations is short.

Let us simply seek the solution in the form of polynomials (Taylor series) for each component. A solution without iterations is possible if the polynomial degree is *q=3*:

$$\boldsymbol{r}(t) = \sum_{i=0}^{3} (t - t_0)^i \cdot \boldsymbol{c}_i$$

t_0 in principle may be selected arbitrarily. We will select it to lie in the center of the interval.

The coefficients will be determined (a) to meet the boundary conditions and to meet the EQs of motion at t_0.

By doing that the condition equations are linear in the coefficients c_i and thus may be easily calculated.

Orbit Determination as Boundayry Value Problem

The system of condition equations reads as follows:

$$(\boldsymbol{r}(t_{b_1})=)\quad \sum_{i=0}^{3}(t_{b_1}-t_0)^i\cdot\boldsymbol{c}_i=\boldsymbol{r}_{b_1}$$

$$(\boldsymbol{r}(t_{b_2})=)\quad \sum_{i=0}^{3}(t_{b_2}-t_0)^i\cdot\boldsymbol{c}_i=\boldsymbol{r}_{b_2}$$

$$\ddot{\boldsymbol{r}}(t_{b_1})=)\quad \sum_{i=2}^{3}i\cdot(i-1)\cdot(t_{b_1}-t_0)^{i-2}\cdot\boldsymbol{c}_i=-\mu\cdot\frac{\boldsymbol{r}_{b_1}}{r^3_{b_1}}$$

$$\ddot{\boldsymbol{r}}(t_{b_2})=)\quad \sum_{i=2}^{3}i\cdot(i-1)\cdot(t_{b_2}-t_0)^{i-2}\cdot\boldsymbol{c}_i=-\mu\cdot\frac{\boldsymbol{r}_{b_2}}{r^3_{b_2}}$$

With given RHSs the above equations may be solved easily.

Varying Δ_1 and Δ_2 systematically, the correct solution minimizes the sum of the squared residuals.

Orbit Determination as Boundayry Value Problem

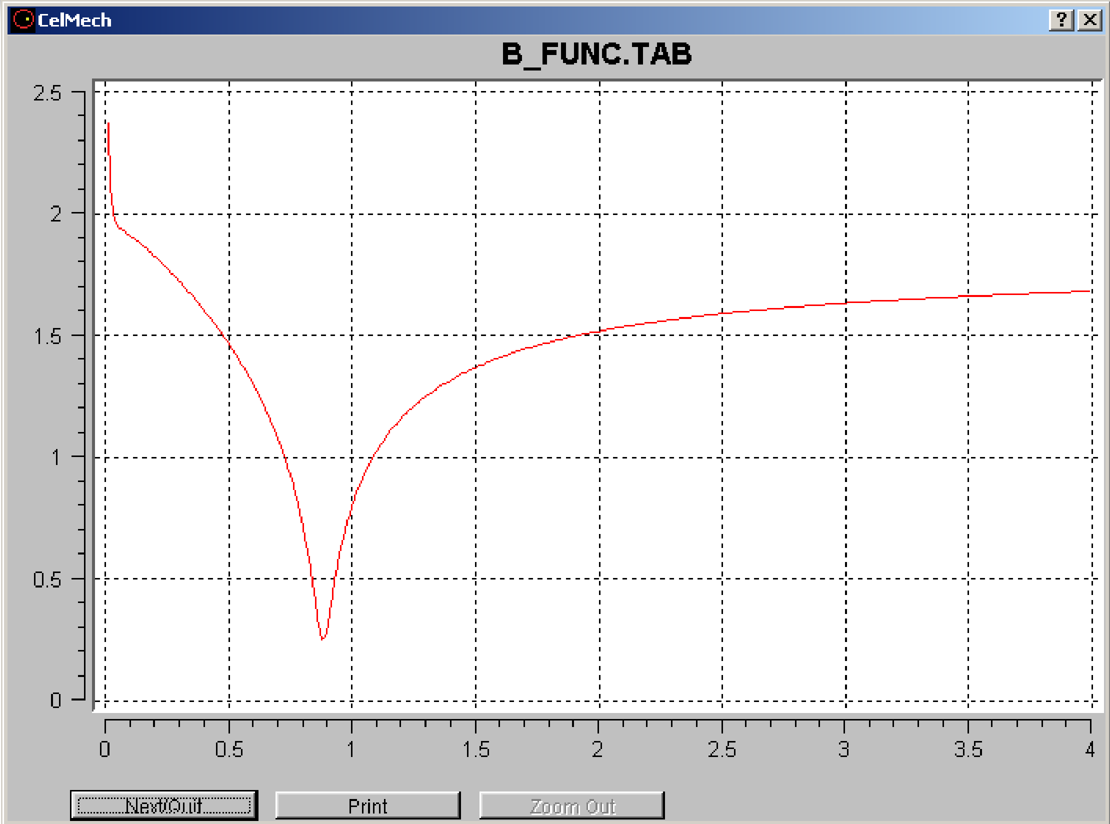
The program system CelMech contains ORBDET which may be used for first orbit determination of minor planets or artificial Earth satellites (or space debris).

As an example the orbit of MP *Silentium* was determined with a special algorithm reducing the problem to a one-dimensional parameter estimation:

- The left boundary vaue Δ_1 is variad systematically. For each seceted value the best possible Δ_2 is determined by least squares (orbit determination with only one parameter).
- The (logarithms) of the mean errors a posteriori of these parameter estimation processes are drawn as a function of Δ_1 .
- The minimum (the minima) of this function are calculated and used as candidate boundary values.

For details consult Beutler (2005), Vol. 1, Sec 8.3.4, first example.

Orbit Determination as Boundayry Value Problem



Observations 1-11 of Silentium were used.

Observations 3 und 11 were selected to define the boundaries.

The figure shows the logarithm of the RMS errors. The minimum at $\Delta_1 = 0.882$ AU is well defined.

Parameter estimation: Variational equations

So far it was assumed that the orbit is adequately described by the TBP, what is usually true when the time interval covered by observations is much shorter than the revolution period of the CB.

For CBs in the planetary system all observations have to refer to one and the same opposition of the CB. For objects in the Earth-near space the observations have to refer to the same observation night (they have to lie within a few minutes for LEOs within about an hour for objects in the GEO).

For orbit improvement need the partial derivatives of the observed functions w.r.t. the orbit parameters. We used the chain rule to write these partial derivatives as a scalar product of the gradient of the observed function w.r.t. the topocentric vector from the observer to the CB and the partial derivative of the geo- or heliocentric position vector of the satellite w.r.t. the orbit parameter.

For an observed function *o* (e.g., angle, distance, coordinate) we put:

$$\frac{\partial o}{\partial I_j} = \sum_{i=1}^{3} \frac{\partial o}{\partial \Delta_i} \cdot \frac{\partial r_i}{\partial I_j}$$

Parameter estimation: Variational equations

We use the same decomposition when the orbit is described by a general EQ of motion (more complicated than that of the TBP).

As the gradient of the observed function will be the same, one only has to deal with the partial derivatives of the orbit $\boldsymbol{r}(t)$ w.r.t. the orbit parameters.

Generalizations are, however, necessary:

- The orbital parameters $I_1, I_2, \ldots, I_6$ have to be replaced by the osculating elements $I_1(t_0), I_2(t_0), \ldots, I_6(t_0)$ at the starting epoch t_0 (or by any set of six parameters uniquely defining the state vector at t_0).
- We have to allow for orbit parameters defining the force model of the general EQs of motion. These parameters are called dynamical.
- We may have to allow for instantaneous velocity changes at predefined epochs.

Parameter estimation: Variational equations

$$\ddot{r} = f(r, \dot{r}, q_1, q_2, ..., q_d)$$

$$r_0 = r(t_0; a_0, e_0, i_0, \Omega_0, \omega_0, T_0)$$

$$v_0 = \dot{r}(t_0; a_0, e_0, i_0, \Omega_0, \omega_0, T_0)$$

$$[p_1, p_2, ..., p_{6+d}] = [a_0, e_0, i_0, \Omega_0, \omega_0, T_0, q_1, q_2, ..., q_d]$$

$$p \in [p_1, p_2, ..., p_{6+d}]$$

Let us assume that a satellite orbit is parameterized by six osculating elements (defining the initial state vector at t_0) and *d* so-called dynamical parameters q_i , defining the force field acting on the satellite. We may think of the q_i as the coefficients of the Earth's gravity field or any other scaling factors of force constituents.

We are interested in the partial derivative of the orbit w.r.t. any of the parameters p_i , $i=1,2,\dots, 6+d$.

Parameter estimation: Variational equations

$$z(t) \doteq \frac{\partial r(t)}{\partial p}$$

$$\ddot{z} = A_0(t) \cdot z(t) + A_1(t) \cdot \dot{z}(t) + \frac{\partial f(t)}{\partial p}$$

$$z(t_0) = \frac{\partial r_0}{\partial p}; \dot{z}(t_0) = \frac{\partial v_0}{\partial p}$$

$$where:$$

$$A_{0,ik} = \frac{\partial f_i}{\partial r_k}; A_{1,ik} = \frac{\partial f_i}{\partial \dot{r}_k}$$

We denote ***z***(*t*) as the partial derivative of the orbit ***r***(*t*) w.r.t. an arbitrarily selected parameter *p*. The variational EQs are obtained by taking the partial derivate of the EQs of motion.

The corresponding initial conditions are obtained by taking the partial derivate w.r.t. to *p* of the corresponding initial values of the EQs of motion. It is straight forward, but may be cumbersome to calculate the elements of the matrices $\boldsymbol{A}_i$, *i=0,1.*

Parameter estimation: Variational equations

$$\ddot{z} = A_0(t) \cdot z(t) + A_1(t) \cdot \dot{z}(t)$$

$$z(t_0) = \frac{\partial r_0}{\partial I_{0,l}}; \dot{z}(t_0) = \frac{\partial v_0}{\partial I_{0,l}}$$

$$\ddot{z} = A_0(t) \cdot z(t) + A_1(t) \cdot \dot{z}(t) + \frac{\partial f}{\partial q}$$

$$z(t_0) = \dot{z}(t_0) = 0$$

The variational equations are linear DEQs.

p = osculating element (left): The variational EQs are homogeneous, the initial state vector is $\neq$ ***0***.

p = dynamical parameter (right): The variational EQs are inhomogeneous, the initial state vector is = ***0***.

The homogeneous part is the same in both types of variational EQs.

Parameter estimation: Variational equations

Characteristics of homogeneous linear DEQs:

Each homogeneous solution may be written as a linear combination of the initial values at t_0.

There are six independent solutions of the homogeneous EQ on the previous page – corresponding to the six osculating elements.

The six solutions are said to form a complete system of solutions.

Any homogeneous solution is a LC of the six independent solutions.

An instantaneous change δv of the velocity vector at a particular epoch t along a user-defined unit vector $\boldsymbol{e}$ may be interpreted as a change in the initial state vector referring to that epoch.

Consequently the partial derivative of the orbit w.r.t. this parameter δv may be written as a linear combination of the six solutions forming the complete system. The coefficients of the LC are constant.

Parameter estimation: Variational equations

Let us assume that we want to allow for an instantaneous velocity change of the orbit ***r***(*t*) at the epoch t_i in the direction of the unit vector ***e***. We want the resulting orbit to be continuous.

The difference of the new – old orbit at t_i obviously is given for $t = t_i$ by:

$$\delta \dot{\boldsymbol{r}}(t_i) = \delta v \, \mathbf{e}$$

$$\delta \boldsymbol{r}(t_i) = \mathbf{0} \; .$$

Let us assume that at the epoch t_i we want to allow for an instantaneous velocity change of the orbit ***r***(*t*) in the direction of the unit vector ***e***. We want the resulting orbit to be continuous.

The difference of the new – old orbit for $t \geq t_i$ obviously is given by:

$$\delta \boldsymbol{r}(t) = \left(\frac{\partial \boldsymbol{r}}{\partial \left(\delta v\right)} \right)(t) \; \delta v$$

Parameter estimation: Variational equations

where:

$$\left(\frac{\partial \boldsymbol{r}}{\partial\left(\delta v\right)}\right)(t_i) = \boldsymbol{0}$$

$$\left(\frac{\partial \dot{\boldsymbol{r}}}{\partial\left(\delta v\right)}\right)(t_i) = \boldsymbol{e}$$

As the partial derivative is a solution of the homogeneous variational equations, we may write

$$\left(\frac{\partial \boldsymbol{r}}{\partial\left(\delta v\right)}\right)(t) = \sum_{k=1}^{6} \beta_k \left(\frac{\partial \boldsymbol{r}}{\partial I_k}\right)(t) \stackrel{\text{def}}{=} \sum_{k=1}^{6} \beta_k \, \boldsymbol{z}_k(t)$$

The time independent coefficients β_k *still* have to be determined.

Parameter estimation: Variational equations

This is, however, easy: we just have to introduce the LC of the six partial derivatives w.r.t. the osculating elements at time t_0 into the equations defining the partial derivatives w.r.t. δv at time t_i:

$$\sum_{k=1}^{6} \beta_k \, z_k(t_i) = 0$$

$$\sum_{k=1}^{6} \beta_k \, \dot{z}_k(t_i) = e$$

Observe that this system can be solved for good and all.

There is one set of coefficients for each pulse. Even if hundreds of pulses are introduced, there is no necessity to solve additional variational equations. All partial derivatives may be found as LC of the six partial derivatives w.r.t. the osculating elements.

Parameter estimation: Variational equations

What about the variational equations associated with the dynamical parameters q_i?

The theory of linear DEQs tells that the solution of an inhomogeneous linear DEQ may be written as a LC of the six homogeneous solutions forming the complete system. The coefficients of the six solutions are, however, time-dependent.

The time-dependent coefficients may be found by quadrature, i.e., as solutions of definite integrals (no longer as the solution of linear DEQ systems).

Details may be found in Beutler (2005), Vol. 1, Sec 5.2, where the problem is solved for a DEQs of order n and dimension d.