Impact of Earth rotation on the signal propagation between CERN and Gran Sasso


G. Beutler, A. Jäggi, T. Schildknecht, R. Dach
Astronomical Institute, University of Bern,
Sidlerstrasse 5, 3012 Bern, Switzerland

1 Introduction

The slant range $d_1$ associated with a signal traveling with speed of light $c$ from $P_1$ at time $t$ to $P_2$ is the distance between $P_1$ at emission time $t$ and $P_2$ at reception time $t + \Delta t$, where $\Delta t = d_1/c$ is the propagation time of the signal between $P_1$ and $P_2$.

$P_1$ and $P_2$ are assumed to be at rest in the Earth-fixed coordinate system and to have the geographic coordinates $r_i, \lambda_i, \phi_i, i = 1, 2$. $d_1$ is calculated in two different ways, which both give the same result.

2 Geographic and Quasi-inertial Coordinates

Let:

$$r_i, \phi_i, \lambda_i, i = 1, 2$$

the spherical geographic coordinates, namely geocentric radius vector, latitude, and longitude, of the signal transmitter at $P_1$ and the signal detector at $P_2$ in the Earth-fixed coordinate system.

The corresponding rectangular coordinates are:

$$r_{Ei} = \begin{pmatrix} x_{Ei} \\ y_{Ei} \\ z_{Ei} \end{pmatrix} = r_i \begin{pmatrix} \cos \phi_i \cos \lambda_i \\ \cos \phi_i \sin \lambda_i \\ \sin \phi_i \end{pmatrix}.$$ (2)

The distance in the Earth-fixed system between $P_1$ and $P_2$ is:

$$d_0 = \sqrt{(x_{E2} - x_{E1})^2 + (y_{E2} - y_{E1})^2 + (z_{E2} - z_{E1})^2}.$$ (3)

Assuming a simplified model for the rotation of the Earth (neglecting polar motion, precession and nutation) the Earth-fixed coordinates may be easily transformed into a quasi-inertial coordinate system, see (Beutler, 2005):

$$r_{Ii} = R_3(-\Theta) r_{Ei}, i = 1, 2,$$ (4)

where $r_{Ii}$ are the coordinates in the quasi-inertial system, $\Theta$ is the Greenwich sidereal time, and $R_3(\alpha)$ describes the particular rotation about the third coordinate axis and angle $\alpha$. The resulting system is called quasi-inertial, because the geocenter, the origin of the system is in accelerated motion about the Sun. Apart from that the coordinate array $r_{Ii}$ refers to the geocentric, equatorial system.

3 Determining the Slant Range $d_1$:

Method 1

3.1 General Relation

The slant range between between $P_1$ and $P_2$ is defined as:

$$d_1 = |r_{I2}(t + \Delta t) - r_{I1}(t)|.$$ (5)

$d_1$ thus is the distance in the quasi-inertial system between $P_1$ at transmission time and $P_2$ at reception time. $\Delta t$ is the signal traveling time. Assuming an experiment in vacuum and neglecting the influence of the geopotential we may approximate:

$$\Delta t = d_0/c,$$ (6)

where $c = 299792458$ m/s is the speed of light in vacuo.

Let us now adopt a special quasi-inertial coordinate system with its coordinate plane $(x_{I1}, x_{I3})$ coinciding with the meridian of the transmitter at emission time $t$. For this special selection we have $\Theta(t) = 0$ and therefore

$$\Theta(t + \Delta t) = \omega \cdot \Delta t,$$ (8)

where $\omega = 7.292115 \cdot 10^{-5}$ rad/s is the angular velocity of Earth rotation. We may thus establish the coordinate transformation for $P_2$ at $t + \Delta t$ into the inertial system as:

$$r_{I2}(t + \Delta t) = \begin{pmatrix} \cos \omega \Delta t - \sin \omega \Delta t & 0 \\ \sin \omega \Delta t & \cos \omega \Delta t \\ 0 & 0 & 1 \end{pmatrix} r_{E2}.$$ (9)
As \( \omega \Delta t \) is a small angle we may use the approximation:

\[
\begin{align*}
  r_{I2}(t + \Delta t) &= \left( \begin{array}{cc}
    1 & -\omega \Delta t \\
    \omega \Delta t & 1 \\
    0 & 0
  \end{array} \right) r_{E2} \\
  \text{or} \\
  r_{I2}(t + \Delta t) &= \left( \begin{array}{c}
    x_{E2} - \omega \Delta t \ y_{E2} \\
    y_{E2} + \omega \Delta t \ x_{E2}
  \end{array} \right). \tag{10}
\end{align*}
\]

Introducing this relation into Eq. (5) and neglecting higher order terms we obtain:

\[
d_1 = d_0 + \omega \Delta t \cdot [x_{E1} y_{E2} - x_{E2} y_{E1}] / d_0. \tag{12}
\]

Using Eq. (2) to further modify the above formula and well-known trigonometric relations gives the final result:

\[
d_1 = d_0 + \omega \Delta t \left\{ \frac{r_{I2}}{d_0} \cos \phi_1 \cos \phi_2 \sin(\lambda_2 - \lambda_1) \right\}. \tag{13}
\]

Formula (13) shows that \( d_1 = d_0 \) for \( \lambda_2 = \lambda_1 \) and that it is maximum if transmitter and detector are located on the equator and separated by a certain longitude difference. It is furthermore clear that \( d_1 > d_0 \) for \( \lambda_2 > \lambda_1 \), i.e., if \( P_2 \) is located in the East of \( P_1 \), we have \( d_1 - d_0 > 0 \), i.e. the slant range is larger than the geometric distance \( d_0 \) between \( P_1 \) and \( P_2 \).

### 3.2 Application to the OPERA Experiment

Let us now calculate the difference \( d_1 - d_0 \), if \( P_1 \) is the CERN in Geneva and \( P_2 \) the Gran Sasso laboratory in Italy. We assume the following spherical Earth-fixed coordinates for \( P_1 \) and \( P_2 \):

\[
\begin{align*}
  P_1 & : r_1 = 6368000 \ m \ \phi_1 = 46.20^\circ \ \lambda_1 = 06.15^\circ \\
  P_2 & : r_2 = 6368000 \ m \ \phi_2 = 42.47^\circ \ \lambda_2 = 13.55^\circ.
\end{align*} \tag{14}
\]

The mean Earth radius was taken for \( r_i, i = 1, 2 \), and rather approximate coordinates (from the internet) for the geographical coordinates of CERN and Gran Sasso. The distance \( d_0 \) corresponding to the adopted and approximate coordinates is \( d_0 = 718800 \ m \). With these values we obtain:

\[
d_1 - d_0 = 0.579 \ m. \tag{15}
\]

This corresponds to a correction of the signal propagation time due to Earth rotation of

\[
\Delta t (\text{Earth rotation}) = \frac{d_1 - d_0}{c} = 1.93 \ \text{ns}. \tag{16}
\]

The result corresponds to the crude coordinates (14). Better coordinates have to be used for generating the final result.

### 4 Determining the Slant Range \( d_1 \): Method 2

We introduce an equatorial Earth-fixed coordinate system, with its zero meridian going through the Gran Sasso laboratory. The spherical coordinates of \( P_1 \) and \( P_2 \) in this system are:

\[
\begin{align*}
  P_1 & : r_1 = 6368000 \ m \ \phi = 46.20^\circ \ \tilde{\lambda}_1 = -7.40^\circ \\
  P_2 & : r_2 = 6368000 \ m \ \phi = 42.47^\circ \ \tilde{\lambda}_2 = 0.00^\circ,
\end{align*} \tag{17}
\]

where \( \tilde{\lambda}_1 = \lambda_1 - \lambda_2 \). The corresponding rectangular coordinates are:

\[
\begin{align*}
  r_{E1} &= r_1 \left( \begin{array}{c}
    \cos \phi \cos \tilde{\lambda}_1 \\
    \cos \phi \sin \tilde{\lambda}_1 \\
    \sin \phi
  \end{array} \right), \\
  r_{E2} &= r_2 \left( \begin{array}{c}
    \cos \phi_2 \\
    0 \\
    \sin \phi_2
  \end{array} \right).
\end{align*} \tag{18}
\]

In analogy to the deliberations in Sect. 3 we introduce the inertial system coinciding with the new Earth-fixed system at time \( t \). In this system the Gran Sasso laboratory has the velocity vector

\[
\mathbf{v}_{I2} = r_2 \omega \left( \begin{array}{c}
  0 \\
  \cos \phi_2 \\
  0
\end{array} \right). \tag{19}
\]

The change of the length of the baseline vector between \( P_1 \) at \( t \) and \( P_2 \) at \( t + \Delta t \) consequently is:

\[
\begin{align*}
  d_1 - d_0 &= \mathbf{v}_{I2} \cdot \{r_{E2} - r_{E1}\} / d_0 \ \Delta t \\
  &= -r_2 \omega \cos \phi_2 \left \{ r_1 \cos \phi_1 \sin \tilde{\lambda}_1 \right \} / d_0 \ \Delta t \\
  &= \omega \Delta t \frac{d_0}{d_0} \cos \phi_1 \cos \phi_2 \sin(\lambda_2 - \lambda_1). \tag{20}
\end{align*}
\]

which is identical to the result (13) obtained with the first method.

**Acknowledgements.** Hans Bebie contributed the independent check in Sect. 4, what is gratefully acknowledged.

**References**